Variations of Mixed Hodge Structure, Higgs Fields, and Quantum Cohomology .

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Abstract

Following C. Simpson, we show that every variation of graded-polarized mixed Hodge structure defined over $\mathbb{Q}$ carries a natural Higgs bundle structure $\bar{\partial} + \theta$ which is invariant under the $\mathbb{C}^*$ action studied in [20]. We then specialize our construction to the context of [6], and show that the resulting Higgs field θ determines (and is determined by) the Gromov-Witten potential of the underlying family of Calabi–Yau threefolds.

1 Introduction

Let X be a compact Kähler manifold. Then, by virtue of [20], it is known that a monodromy representation

$$\rho : \pi_1(X, x_0) \to GL(V) \tag{1.1}$$

arises from a variation of pure, polarized Hodge structure $\mathcal{V} \to X$ if and only if

(1) ρ is semisimple.

(2) The associated Higgs bundle structure $\bar{\partial}+\theta$ is invariant under the $\mathbb{C}^*$ action (1.9).

(3) The representation ρ is defined over $\mathbb{R}$ relative to some choice of real structure $V_{\mathbb{R}}$ on V.

To study the non-semisimple representations of $\pi_1(X, x_0)$, we begin with a variation of graded-polarized mixed Hodge structure

$$\mathcal{V} \to X$$

and ask whether the underlying C^∞ vector bundle of $\mathcal{V}$ carries a natural Higgs bundle structure $\bar{\partial} + \theta$ invariant under the $\mathbb{C}^*$ action (1.9) studied in [20]. Our main result is as follows:

Theorem 5.1 *Let $V \to S$ be a variation of graded-polarized mixed Hodge structure, and $\{\mathcal{U}^p\}$ denote the collection of C^∞ subbundles of $\mathcal{V}$ defined by the rule:*

$$\mathcal{U}^p_s = \bigoplus_q I^{p,q}_{(\mathcal{F}_s, \mathcal{W}_s)}$$

Then, relative to the Gauss–Manin connection ∇, the direct sum decomposition

$$\mathcal{V} = \bigoplus_p \mathcal{U}^p$$

defines a (unpolarized) complex variation of Hodge structure.

Therefore, by virtue of [20], the underlying C^∞ vector bundle E of $\mathcal{V}$ does indeed carry a natural Higgs bundle structure $\bar{\partial} + \theta$. Moreover, because the Higgs bundle structure $\bar{\partial} + \theta$ produced by Theorem (5.1) arises from a complex variation of Hodge structure, it is automatically a fixed point of (1.9). [See Lemma (1.12) for details.]

Remark. *In the case of variations of graded-polarized mixed Hodge structure arising from the cohomology of a family of singular or quasi-projective varieties, the corresponding Higgs field θ produced by Theorem* (5.1) *turns out to be a natural analog of the Kodaira–Spencer map associated to a smooth family of non-singular projective varieties.*

The main tools used in proof of Theorem (5.1) are Deligne's observation that every mixed Hodge structure (F, W) determines a functorial bigrading

$$V = \bigoplus_{p,q} I^{p,q}_{(F,W)}$$

of the underlying vector space V which mimics the classical Hodge decomposition (although in general $\bar{I}^{p,q} \neq I^{q,p}$), and the classifying spaces $\mathcal{M}$ of graded-polarized mixed Hodge structure described in §3.

Remark. *An alternative proof of Theorem* (5.1) *due to P. Deligne [7] is outlined in the appendix.*

The general outline of this paper is as follows: After presenting some basic definitions at the end of this section and the preliminary remarks of §2, we begin §3 with a review of the classifying spaces $\mathcal{D}$ of pure, polarized Hodge structure constructed in [10]. Following [13], we then define classifying space of graded-polarized mixed Hodge structure which are universal with respect to variations of graded-polarized mixed Hodge structure.

In analogy with the pure case, the fundamental C^∞ vector bundles supported by such classifying spaces are the turn out to be the Deligne–Hodge bundles

$$\mathcal{I}^{p,q}_F = I^{p,q}_{(F,W)}$$

These bundles are studied in §4, wherein we extend the methods of [5] to obtain an explicit formula governing the first order behavior of the decomposition:

$$V = \bigoplus_{p,q} \mathcal{I}^{p,q}_{(F,W)}$$

along $\mathcal{M}$. Careful application of this formula then gives the proof of Theorem (5.1) presented at the beginning of §5.

To study the asymptotic behavior of such Higgs bundles, we then prove a "group theoretic" version of Schmid's Nilpotent Orbit Theorem [Formula (6.8)] and derive an equivalence of categories theorem for unipotent variations of mixed Hodge structure [Theorem (6.19)].

Armed with these results, we then specialize our constructions to the context of mirror symmetry and quantum cohomology in §7 and §8. In doing so, we obtain interpretations of both Deligne's work on the local behavior of Hodge structures at infinity [6] and some recent work of David Cox and Sheldon Katz [4] in terms of Higgs fields associated to variations of graded-polarized mixed Hodge structures.

To close this section, we shall now recall several basic definitions and constructions which are used throughout this paper. Additional background material may be found in §2.

Definition 1.2 *Let S be a complex manifold. Then, following [21], we define a variation $\mathcal{V} \to S$ of graded-polarized mixed Hodge structure to consist of a $\mathbb{Q}$-local system $\mathcal{V}_{\mathbb{Q}}$ defined over S equipped with:*

(1) *A rational, increasing weight filtration*

$$0 \subseteq \cdots \mathcal{W}_k \subseteq \mathcal{W}_{k+1} \subseteq \cdots \subseteq \mathcal{V}_{\mathbb{C}}$$

of $\mathcal{V}_{\mathbb{C}} = \mathcal{V}_{\mathbb{Q}} \otimes \mathbb{C}$.

(2) *A decreasing Hodge filtration*

$$0 \subseteq \cdots \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \subseteq \cdots \subseteq \mathcal{V}_{\mathbb{C}} \otimes \mathcal{O}_S$$

(3) *A collection of rational, non-degenerate bilinear forms*

$$\mathcal{S}_k : Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}) \otimes Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}) \to \mathbb{Q}$$

of alternating parity $(-1)^k$.

satisfying the following mutual compatibility conditions:

(a) *Relative to the Gauss–Manin connection of $\mathcal{V}$:*

$$\nabla \mathcal{F}^p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$$

for each index p.

(b) *The triple $(Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}), \mathcal{F}\, Gr^{\mathcal{W}}_k(\mathcal{V}_{\mathbb{Q}}), \mathcal{S}_k)$ defines a variation of pure, polarized Hodge structure for each index k.*

Definition 1.3 *A Higgs bundle* $(E, \bar\partial + \theta)$ *consists of a holomorphic vector bundle* $(E, \bar\partial)$ *endowed with a endomorphism valued 1-form*

$$\theta : \mathcal{E}^0(E) \to \mathcal{E}^{1,0}(E)$$

which is both holomorphic and symmetric (i.e. $\bar\partial\theta = 0$ *and* $\theta \wedge \theta = 0$*).*

Example 1.4 *Let* $\mathcal{V}$ *denote a variation of pure, polarized Hodge structure arising via the cohomology of a smooth family of non-singular projective varieties* $f : Y \to X$*. Then, by virtue of the* C^∞ *decomposition*

$$\mathcal{V} = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \tag{1.5}$$

underlying smooth vector bundle

$$E = \mathcal{V}_{\mathbb{C}} \otimes \mathcal{E}^0_X$$

of $\mathcal{V}$ *inherits an integrable complex structure* $\bar\partial$ *via the isomorphism*

$$\mathcal{H}^{p,q} \cong \mathcal{F}^p / \mathcal{F}^{p+1}$$

and the holomorphic structure of $\mathcal{F}^p$*. Likewise, the Kodaira–Spencer map*

$$\kappa_p : T_p(X) \to H^1(Y_p, \Theta(Y_p))$$

defines a symmetric, endomorphism valued 1-form θ *on* E *via the rule*

$$\theta(\xi)(\sigma) = \kappa(\xi) \cup \sigma$$

To prove that $(E, \bar\partial + \theta)$ *is indeed a Higgs bundle, observe that by virtue of [10], we may write the Gauss–Manin connection* ∇ *as*

$$\nabla = \tau + \bar\partial + \partial + \theta \tag{1.6}$$

relative to a pair of differential operators

$$\bar\partial : \mathcal{E}^0(E) \to \mathcal{E}^{0,1}(E), \qquad \partial : \mathcal{E}^0(E) \to \mathcal{E}^{1,0}(E) \tag{1.7}$$

preserving the Hodge decomposition (1.5) *and a pair of tensor fields*

$$\begin{array}{l} \tau : \mathcal{H}^{p,q} \to \mathcal{E}^{0,1} \otimes \mathcal{H}^{p+1,q-1} \\ \theta : \mathcal{H}^{p,q} \to \mathcal{E}^{1,0} \otimes \mathcal{H}^{p-1,q+1} \end{array} \tag{1.8}$$

shifting the indices of (1.5) *by* ± 1*. Expanding out the integrability condition* $\nabla^2 = 0$*, and taking account of equations* (1.6)–(1.8)*, it then follows that*

$$\bar\partial^2 = 0, \qquad \bar\partial\theta = 0, \qquad \theta \wedge \theta = 0$$

and hence $(E, \bar\partial + \theta)$ *is a Higgs bundle by virtue of the Newlander-Nirenburg theorem. Moreover, given any element* $\lambda \in \mathbb{C}^*$*, the map*

$$f : \mathcal{V} \to \mathcal{V}, \qquad f|_{\mathcal{H}^{p,q}} = \lambda^p$$

defines a bundle isomorphism

$$(E, \bar\partial + \theta) \cong (E, \bar\partial + \lambda\theta)$$

Consequently, the isomorphism class of such a Higgs bundle $(E, \bar\partial + \theta)$ *is a fixed point of the* $\mathbb{C}^*$ *action*

$$\lambda : (E, \bar\partial + \theta) \to (E, \bar\partial + \lambda\theta) \tag{1.9}$$

In order to construct deformations of such Higgs bundles, Simpson introduces the following definition:

Definition 1.10 *A complex variation of Hodge structure consists of the following data:*

(1) *A flat,* $\mathbb{C}$*-vector bundle* (E, ∇).

(2) *A* C^∞ *decomposition*

$$E = \bigoplus_p \mathcal{U}^p \tag{1.11}$$

satisfying Griffiths' horizontality, i.e.

$$\nabla : \mathcal{E}^0(\mathcal{U}^p) \to \mathcal{E}^{0,1}(\mathcal{U}^{p+1}) \oplus \mathcal{E}^{0,1}(\mathcal{U}^p) \oplus \mathcal{E}^{1,0}(\mathcal{U}^p) \oplus \mathcal{E}^{1,0}(\mathcal{U}^{p-1})$$

Remark. *A flat hermitian form* Q *is said to polarize the complex variation* $(E, \nabla, \oplus_p \mathcal{U}^p)$ *provided bilinear form*

$$\langle u, v \rangle = Q(Cu, v), \qquad C|_{\mathcal{U}^p} = (-1)^p$$

is positive definite and makes the direct sum decomposition (1.11) *In general, the complex variations of Hodge structure considered in this paper will be unpolarized.*

Lemma 1.12 *Every complex variation of Hodge structure* $(E, \nabla, \mathcal{U}^*)$ *carries a natural Higgs bundle structure* $\bar\partial + \theta$ *invariant under the* $\mathbb{C}^*$ *action* (1.9).

Proof. One simply goes through the proof presented in Example (1.4), replacing $\mathcal{H}^{p,q}$ with $\mathcal{U}^p$.

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2 Preliminary Remarks

The purpose of this section is to acquaint the reader with P. Deligne's theory of mixed Hodge structures, and provide a catalog of basic definitions for later use. We assume only that the reader is already familiar with the basic tenets of Hodge theory, as outlined in [11].

Conceptually, a the notion of a mixed Hodge structure may be viewed as a kind of "iterated extension" of pure Hodge structures. Thus, as a prelude to the formal definition presented below, let us first consider the problem of defining what it should mean for an exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{2.1}$$

to define an extension of pure Hodge structures. Since both A and C are filtered vector spaces, one might be tempted to require only that the vector space $B = B_{\mathbb{Q}} \otimes \mathbb{C}$ carry a decreasing "Hodge filtration" $F^{\bullet}(B)$ which is strictly compatible with the given maps α and β, i.e.

$$\alpha(F^p(A)) = F^p(B) \cap \alpha(A), \quad \beta(F^p(B)) = F^p(C) \cap \beta(B) \tag{2.2}$$

The trouble with this preliminary definition is that it does not encode the weights of the pure Hodge structures A and C. To rectify this defect, observe that a pure Hodge structure of weight k may be completely recovered from the knowledge of its Hodge filtration F and its weight filtration

$$W_j(V_{\mathbb{Q}}) = \begin{cases} V_{\mathbb{Q}} & j \geq k \\ 0 & j < k \end{cases}$$

via the rule $H^{p,q} = F^p \cap \bar{F}^q \cap W_{p+q}$. In light of this observation, it is therefore natural to require that B carry an increasing "weight filtration"

$$0 \subseteq \cdots \subseteq W_{k-1}(B_{\mathbb{Q}}) \subseteq W_k(B_{\mathbb{Q}}) \subseteq \cdots \subseteq B_{\mathbb{Q}}$$

which is strictly compatible with the given maps α and β, i.e.

$$\begin{aligned} \alpha(W_k(A_{\mathbb{Q}})) &= W_k(B_{\mathbb{Q}}) \cap \alpha(A_{\mathbb{Q}}) \\ \beta(W_k(B_{\mathbb{Q}})) &= W_k(C_{\mathbb{Q}}) \cap \beta(B_{\mathbb{Q}}) \end{aligned} \tag{2.3}$$

To see that these two requirements define a reasonable notion of what it should mean for B to represent an extension of C by A, observe that one at least has the following result:

Lemma 2.4 *Let $F^\bullet(B)$ and $W_\bullet(B_{\mathbb{Q}})$ be a pair of filtrations which satisfy condition (2.2) and (2.3). Then, for each index k, the "Hodge filtration" $F^\bullet(B)$ induces a pure Hodge structure of weight k on the quotient space $Gr_k^{W(B)} := W_k(B)/W_{k-1}(B)$ via the rule:*

$$F^p Gr_k^{W(B)} = \frac{F^p(B) \cap W_k(B) + W_{k-1}(B)}{W_{k-1}(B)}$$

The pair $(F(B), W(B))$ constructed above is an example of a mixed Hodge structure. The geometric importance such structures rests upon P. Deligne's construction of functorial mixed Hodge structures on the cohomology of an arbitrary complex algebraic variety X.

Definition 2.5 *Let $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ be a finite dimensional complex vector space defined $\mathbb{Q}$. Then, a decreasing filtration F of V is said to pair with an increasing filtration $W = W(V_{\mathbb{Q}}) \otimes \mathbb{C}$ to define a mixed Hodge structure (F, W) provided they satisfy the following condition: For each index k, the induced filtration*

$$F^p Gr_k^W = \frac{F^p \cap W_k + W_{k-1}}{W_{k-1}}$$

defines a pure Hodge structure of weight k on quotient space

$$Gr_k^W = W_k/W_{k-1}$$

Example 2.6 *Let $\mathbb{S}$ be a finite set of distinct points in a compact Riemann surface M and $\Omega^1_M(\mathbb{S})$ denote the space of meromorphic 1-forms on M which have at worst simple poles along $\mathbb{S}$. Then, the mixed Hodge structure (F, W) attached to $H^1(M \setminus \mathbb{S}, \mathbb{C})$ by Deligne's construction is given by the following pair of filtrations:*

$$\begin{array}{lll} W_0 = 0 & W_1 = H^1(M, \mathbb{C}) & W_2 = H^1(M \setminus \mathbb{S}, \mathbb{C}) \\ F^2 = 0 & F^1 = \Omega^1_M(\mathbb{S}) & F^0 = H^1(M \setminus \mathbb{S}, \mathbb{C}) \end{array}$$

To obtain an analog of the Hodge decomposition in the category of mixed Hodge structures, Deligne proceeds as follows:

Definition 2.7 *A bigrading of a mixed Hodge structure (F, W) is a direct sum decomposition $V = \bigoplus_{p,q} J^{p,q}$ of the the underlying complex vector space V which has the following two properties:*

- $F^p = \oplus_{r \geq p, s} J^{r,s}$.
- $W_k = \oplus_{r+s \leq k} J^{r,s}$.

Lemma 2.8 *Let (F, W) be a mixed Hodge structure. Then, there exists a unique bigrading $\{I^{p,q}\}$ of (F, W) with the following additional property:*

$$I^{p,q} = \bar{I}^{q,p} \mod \bigoplus_{r<p, s<q} I^{r,s}$$

Example 2.9 *In the case of the finitely punctured Riemann surface $M \setminus \mathbb{S}$ considered previously, the Deligne–Hodge decomposition of (F, W) is given by the following subspaces of $H^1(M \setminus \mathbb{S}, \mathbb{C})$:*

$$I^{1,1} = F^1 \cap \overline{F^1}, \qquad I^{1,0} = H^{1,0}(M), \qquad I^{0,1} = H^{0,1}(M)$$

Moreover, in this particular case, the subspace $I^{1,1} \subseteq F^1$ admits the following description: Let $\mathcal{H}^0_M(\mathbb{S})$ denote the space of real-valued harmonic functions on M which have at worst logarithmic singularities along $\mathbb{S}$. Then,

$$I^{1,1} \cap H^1(M \setminus \mathbb{S}, \mathbb{R}) = \left\{ \sqrt{-1}\,\frac{\partial f}{\partial z} dz \mid f \in \mathcal{H}^0_M(\mathbb{S}) \right\}$$

Remark. **[Kaplan]** *By virtue of Lemma* (2.8)*, the subspaces*

$$\Lambda^{p,q}(V) := \bigoplus_{a \leq p, b \leq q} I^{p,q}$$

satisfy the symmetry condition $\overline{\Lambda^{p,q}(V)} = \Lambda^{q,p}(V)$. In particular,

$$\bar{I}^{p,q} = I^{q,p} \mod \Lambda^{q-1,p-1}(V) \tag{2.10}$$

Regarding the functorial properties of mixed Hodge structures, one has the following basic result:

Theorem 2.11 *The category of mixed Hodge structures defined over a fixed subfield $R \subseteq \mathbb{R}$ is abelian. Moreover, it is closed under the operations of taking direct sums, tensor products, and duals.*

Corollary 2.12 *The choice of a mixed Hodge structure (F, W) on $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ induces a mixed Hodge structure on $gl(V)$ via the bigrading:*

$$gl(V)^{r,s} = \{\, \alpha \in gl(V) \mid \alpha : I^{p,q} \to I^{p+r,q+s} \quad \forall\, p, q \,\}$$

Remark. *A morphism of mixed Hodge structure $f : V \to V'$ is a R-linear map which is strictly compatible with the filtrations F and W.*

Definition 2.13 *A graded-polarization of a mixed Hodge structure (F, W) consists of a choice of polarization $\mathcal{S}_k$ for each non-trivial layer Gr^W_k of Gr^W.*

Example 2.14 *Given a finitely puncture Riemann surface $M \setminus \mathbb{S}$, the mixed Hodge structures attached to $H^1(M \setminus \mathbb{S}, \mathbb{C})$ is graded-polarized by the nondegenerate bilinear forms*

$$S_2(\alpha, \beta) = 4\pi^2 \sum_{p \in \mathbb{S}} Res_p(\alpha) Res_p(\beta), \qquad S_1(\alpha, \beta) = \int_M \alpha \wedge \beta$$

defined on Gr^W_2 and Gr^W_1 respectively.

Remark. *Let $\mathcal{S}$ be a graded-polarization of the mixed Hodge structure (F, W) and*

$$G_{\mathbb{C}} = \{\, g \in GL(V)^W \mid Gr(g) \in Aut_{\mathbb{C}}(\mathcal{S})\,\}$$

denote group of automorphisms of V which preserve W and act on Gr^W by infinitesimal isometries. Then, by functoriality, (F, W) determines an induced mixed Hodge structure on $\mathfrak{g} = Lie(G_{\mathbb{C}})$ via the bigrading:

$$\mathfrak{g}^{r,s} := gl(V)^{r,s} \cap Lie(G_{\mathbb{C}}) \tag{2.15}$$

N.b. By virtue of equation (2.10), $\overline{\mathfrak{g}^{r,s}} = \mathfrak{g}^{s,r} \mod \oplus_{a<s,b<r}\, \mathfrak{g}^{a,b}$. Also, $r+s>0 \implies \mathfrak{g}^{r,s}=0$.

To finish our review of Deligne's theory of mixed Hodge structures, we shall recall the basic properties of the weight filtration W and the definition of the relative weight filtration

$${}^rW = {}^rW(N, W)$$

Definition 2.16 *Let W be an increasing filtration of a finite dimensional complex vector space V. Then, a semi-simple endomorphism $Y \in gl(V)$ is said to grade W provided that, for each index k*

$$W_k = W_{k-1} \oplus E_k(Y)$$

(i.e. W_k is the direct sum of W_{k-1} and the k-eigenspace of Y).

Example 2.17 *Let (F, W) be a mixed Hodge structure. Then, by definition, the semi-simple endomorphism Y defined by the rule*

$$Y(v) = kv \iff v \in \bigoplus_{p+q=k} I^{p,q}$$

is a grading of W.

To describe the structure of the set of all gradings of a fixed filtration W, let Lie_{-1} denote the nilpotent ideal of $gl(V)$ defined by the rule:

$$\alpha \in Lie_{-1} \iff \alpha : W_k \to W_{k-1} \;\; \forall\, k \tag{2.18}$$

Theorem 2.19 *The unipotent Lie group $\exp(Lie_{-1})$ acts simply transitively upon the set of all gradings Y of a fixed, increasing filtration W.*

Proof. See [3].

As a prelude to our discussion of the relative weight filtration, let us consider first a more classical object, namely the monodromy weight filtration:

Theorem 2.20 *Let V be a finite dimensional vector space and N be a nilpotent endomorphism of V. Then, there exists a unique* monodromy weight filtration

$$0 \subset W(N)_{-k} \subseteq W(N)_{1-k} \subseteq \cdots \subseteq W(N)_{k-1} \subseteq W(N)_k = V$$

of V with the following two properties:

- $N : W(N)_j \to W(N)_{j-2}$ *for each index j.*
- *The induced maps* $N^j : Gr_j^{W(N)} \to Gr_{-j}^{W(N)}$ *are isomorphisms.*

Example 2.21 *Let ρ be a finite dimensional representation of $sl_2(\mathbb{C})$ and*

$$N_\pm = \rho(n_\pm), \qquad Y = \rho(y)$$

denote the images of the standard generators (n_-, y, n_+) of $sl_2(\mathbb{C})$. Then, by virtue of the semi-simplicity of $sl_2(\mathbb{C})$ and the commutator relations

$$[Y, N_\pm] = \pm 2N_\pm, \qquad [N_+, N_-] = Y$$

it follows that:

$$W(N_-)_k = \bigoplus_{j \leq k} E_j(Y)$$

Definition 2.22 *Given an increasing filtration W of a finite dimensional vector space V and an integer $\ell \in \mathbb{Z}$ the corresponding shifted object $W[\ell]$ is the increasing filtration of V defined by the rule:*

$$W[\ell]_j = W_{j+\ell}$$

Theorem 2.23 *Let W be an increasing filtration of a finite dimensional vector space V. Then, given a nilpotent endomorphism $N : V \to V$ which preserves W, there exists at most one increasing filtration*

$$^rW = {}^rW(N, W)$$

with the following two properties:

- *For each index j,* $N : {}^rW_j \to {}^rW_{j-2}$
- *For each index k, ^rW induces on Gr_k^W the corresponding shifted monodromy weight filtration*

$$W(N : Gr_k^W \to Gr_k^W)[-k]$$

3 Classifying Spaces

In this section, we construct classifying spaces of graded-polarized mixed Hodge structures $\mathcal{M}$ which generalize Griffiths classifying spaces $\mathcal{D}$ of pure, polarized Hodge structures. In particular, we show that any variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ admits a reformulation in terms of its monodromy representation

$$\rho : \pi_1(S, s_0) \to Aut(\mathcal{V}_{s_0})$$

and its period map

$$\phi : S \to \mathcal{M}/\Gamma, \qquad \Gamma = \mathrm{Image}(\rho)$$

Remark. *Classifying spaces of graded-polarized mixed Hodge structures have been studied before, notably in [22] and the unpublished work [13]. For the most part, the presentation given here follows [13].*

To establish notation, let us first review Griffiths construction: Let V be a finite dimensional complex vector space endowed with a rational structure $V_{\mathbb{Q}}$ and a non-degenerate bilinear form

$$Q : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$$

of parity $(-1)^k$. Then, given any partition of $\dim V$ into a sum of non-negative integers $\{h^{p,k-p}\}$, one can from the corresponding classifying space

$$\mathcal{D} = \mathcal{D}(V, Q, h^{p,k-p})$$

consisting of all pure Hodge structure of weight k on V which are polarized by Q and satisfy

$$\dim H^{p,k-p} = h^{p,k-p}$$

A priori, the classifying space $\mathcal{D}$ is just a set. To endow it with the structure of a complex manifold, one may proceed as follows: Let

$$f^p = \sum_{r \geq p} h^{r,k-r}$$

and $\check{\mathcal{F}}$ denote the flag variety consisting of all decreasing filtrations F of V such that

$$\dim F^p = f^p$$

Because $\check{\mathcal{F}}$ is an smooth algebraic manifold, the subset $\check{\mathcal{D}} \subseteq \check{\mathcal{F}}$ consisting of the those filtrations $F \in \check{\mathcal{F}}$ which satisfy the first Riemann bilinear relation

$$Q(F^p, F^{k-p+1}) = 0$$

is also algebraic.

Now, as can be easily checked via elementary linear algebra, the complex Lie group $G_{\mathbb{C}} = Aut_{\mathbb{C}}(Q)$ acts transitively on $\check{\mathcal{D}}$. Consequently, $\check{\mathcal{D}}$ is in fact a smooth subvariety of $\check{\mathcal{F}}$.

To prove that $\mathcal{D}$ is an open subset of $\check{\mathcal{D}}$, observe that because $G_{\mathbb{C}}$ acts transitively on $\check{\mathcal{D}}$, the map

$$g \in G_{\mathbb{C}} \mapsto g.F \in \check{\mathcal{D}}$$

defines a holomorphic surjection from a neighborhood of $1 \in G_{\mathbb{C}}$ onto a neighborhood of $F \in \check{\mathcal{D}}$. Consequently, by virtue of the following lemma, there exists an open subset of $\check{\mathcal{D}}$ about each point $F \in \mathcal{D}$ which is entirely contained in $\mathcal{D}$:

Lemma 3.1 *The Lie group $G_{\mathbb{R}} = Aut_{\mathbb{R}}(Q)$ acts transitively on $\mathcal{D}$. Moreover, given point $F \in \mathcal{D}$, there exists a neighborhood O of $1 \in G_{\mathbb{C}}$ such that*

$$g_{\mathbb{C}} \in O \implies g_{\mathbb{C}}.F \in \mathcal{D} \tag{3.2}$$

Proof. The proof that $G_{\mathbb{R}}$ acts transitively on $\mathcal{D}$ is an exercise in elementary linear algebra which shall be left to the reader.

To verify equation (3.2), observe that (in the notation of §2)

$$Lie(G_{\mathbb{C}}) = \bigoplus_p \mathfrak{g}^{p,-p} \tag{3.3}$$

while the Lie algebra of the isotopy group $G_{\mathbb{C}}^F$ is given by the formula:

$$Lie(G_{\mathbb{C}}^F) = \bigoplus_{p \geq 0} \mathfrak{g}^{p,-p} \tag{3.4}$$

Moreover, the subalgebra

$$Lie(G_{\mathbb{R}}^F) = Lie(G_{\mathbb{C}}^F) \cap Lie(G_{\mathbb{R}}) \subset Lie(G_{\mathbb{C}}^F)$$

consists of exact those elements $\alpha \in \mathfrak{g}^{0,0}$ which are self-conjugate. Consequently,

$$\mathcal{C} = \left(\bigoplus_{p>0} \mathfrak{g}^{p,-p} \right) \bigoplus \sqrt{-1}\, Lie(G_{\mathbb{R}}^F)$$

is a vector space complement to $Lie(G_{\mathbb{R}})$ in $Lie(G_{\mathbb{C}})$, i.e.

$$Lie(G_{\mathbb{C}}) = Lie(G_{\mathbb{R}}) \oplus \mathcal{C} \tag{3.5}$$

By virtue of this vector space decomposition, there exists a neighborhood U_0 of zero in $Lie(G_{\mathbb{C}})$ such that every element

$$g_{\mathbb{C}} \in \exp(U_0)$$

may be uniquely decomposed into a product

$$g_{\mathbb{C}} = g_{\mathbb{R}} g_{\mathbb{C}}^F$$

of an element $g_{\mathbb{R}} \in G_{\mathbb{R}}$ and an element $g_{\mathbb{C}}^F \in \exp(\mathcal{C})$. In particular,

$$g_{\mathbb{C}} \in \exp(U_0) \implies g_{\mathbb{C}}.F = g_{\mathbb{R}} g_{\mathbb{C}}^F.F = g_{\mathbb{R}}.F \in \mathcal{D}$$

Remark. *A smooth map $F : S \to \mathcal{D}$ is holomorphic provided that relative to any choice of local holomorphic coordinates $(s_1, \dots, s_n)$ on S, one has*

$$\frac{\partial F^p}{\partial \bar{s}_j} \subseteq F^p(s_1, \dots, s_n)$$

A holomorphic map $F : S \to \mathcal{D}$ is said to be horizontal provided

$$\frac{\partial F^p}{\partial s_j} \subseteq F^{p-1}(s_1, \dots, s_n)$$

The relationship between variations of pure, polarized Hodge structure (VHS) and the classifying spaces $\mathcal{D}$ is as follows: Let $\mathcal{V} \to S$ be a variation of pure, polarized Hodge structure. Then, choice of base point $s_0 \in S$ determines a monodromy representation

$$\rho : \pi_1(S, s_0) \to \Gamma \tag{3.6}$$

and a (locally-liftable) holomorphic, horizontal map

$$\phi : S \to \mathcal{D}/\Gamma \tag{3.7}$$

via parallel translation of the data of $\mathcal{V}$ to the reference fiber $V = V_{s_0}$.

Conversely, given the monodromy representation (3.6) and the period map (3.7), it is possible to reconstruct the original variation $\mathcal{V} \to S$ (up to isomorphism) by reversing the preceding construction.

To reformulate the notion of a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ in terms of the monodromy representation ρ of $\mathcal{V}$ and a suitable period map $\phi : S \to \mathcal{M}/\Gamma$, one must first construct a suitable classifying space $\mathcal{M}$ of graded-polarized mixed Hodge structures.

To this end, let V be a complex vector space endowed with a choice of rational structure $V_{\mathbb{Q}}$ and a choice of weight filtration W (also defined over $\mathbb{Q}$). Then, given a collection of rational, non-degenerate bilinear forms

$$\mathcal{S}_k : Gr_k^W \otimes Gr_k^W \to \mathbb{C}$$

of alternating parity $(-1)^k$ and a partition of $\dim V$ into suitable sum of non-negative integers $\{h^{p,q}\}$, one can form the classifying space $\mathcal{M}$ consisting of all mixed Hodge structures (F, W) which are graded-polarized by $\mathcal{S}$ and satisfy the dimensionality condition

$$\dim I^{p,q}_{(F,W)} = h^{p,q}$$

In analogy with the pure case, in order to prove that $\mathcal{M}$ is a complex manifold one starts with the flag variety $\check{\mathcal{F}}$ consisting of all decreasing filtrations F of V such that

$$\dim F^p = f^p, \qquad f^p = \sum_{r \geq p, s} h^{r,s}$$

Next, one defines $\check{\mathcal{F}}(W)$ to be the submanifold of $\check{\mathcal{F}}$ consisting of those filtrations which have the following additional property:

$$\dim F^p Gr^W_k = f^p_k, \qquad f^p_k = \sum_{r \geq p} h^{r,k-r}$$

To prove that $\check{\mathcal{F}}(W)$ is a smooth submanifold of $\check{F}$, one simply checks that the complex Lie group

$$GL(V)^W = \{\, g \in GL(V) \mid g : W_k \to W_k \quad \forall k \,\}$$

acts transitively on $\check{\mathcal{F}}(W)$.

To proceed further, one introduces the "compact dual" $\check{\mathcal{M}}$ consisting of all filtrations $F \in \check{\mathcal{F}}(W)$ which satisfy the first Riemann bilinear relation

$$\mathcal{S}_k(F^p Gr^W_k, F^{k-p+1} Gr^W_k) = 0$$

for each index k. As in the pure case, in order to show that $\check{\mathcal{M}}$ is a smooth submanifold of $\check{\mathcal{F}}(W)$, one proves that a suitable Lie group acts transitively on $\check{\mathcal{M}}$:

Lemma 3.8 *The complex Lie group*

$$G_{\mathbb{C}} = \{\, g \in GL(V)^W \mid Gr(g) \in Aut_{\mathbb{C}}(\mathcal{S}) \,\}$$

acts transitively on $\check{\mathcal{M}}$.

Proof. As in the pure case, Lemma (3.8) can be checked by brute force using only elementary linear algebra. Alternatively, one can proceed as follows: Let $\mathcal{Y}(W)$ denote the set of all gradings of W,

$$\check{\mathcal{D}} = \bigoplus_k \check{\mathcal{D}}(Gr^W_k, \mathcal{S}_k, h^{p,k-p})$$

and $\check{\mathcal{X}}$ denote the product space $\check{\mathcal{D}} \times \mathcal{Y}(W)$. Next, let

$$\pi : \check{\mathcal{X}} \to \check{\mathcal{M}}$$

denote the natural projection map which sends a point $(\oplus_k F_k, Y)$ in $\check{\mathcal{X}}$ to the filtration $F \in \check{\mathcal{M}}$ determined by the given filtration $\oplus_k F^{\cdot}_k$ of Gr^W and the induced isomorphism $Y : Gr^W \to V$, i.e.

$$F^p = \bigoplus_k Y(F^p_k)$$

Now, as may be easily checked, the map $\pi : \check{\mathcal{X}} \to \check{\mathcal{M}}$ is both surjective and $G_{\mathbb{C}}$ equivariant. Therefore, in order to prove that $G_{\mathbb{C}}$ acts transitively on $\check{\mathcal{M}}$, it will suffice to prove that $G_{\mathbb{C}}$ acts transitively on $\check{\mathcal{X}}$. To prove the latter assertion, observe that given a point $x = (\oplus_k F_k, Y) \in \check{\mathcal{X}}$, the corresponding isotopy group $G_{\mathbb{C}}^Y$ acts transitively on $\check{\mathcal{D}}$ while fixing the given grading Y. On the other hand, the unipotent Lie group $\exp(Lie_{-1})$ discussed in §2 acts simply transitively on $\mathcal{Y}(W)$ while leaving $\mathcal{D}$ pointwise fixed.

In analogy with the pure case, the proof of the fact that $\mathcal{M}$ is an open subset of $\check{\mathcal{M}}$ follows directly from that fact that $G_{\mathbb{C}}$ acts transitively on $\check{\mathcal{M}}$ together with the following lemma:

Lemma 3.9 *The Lie group*

$$G = \{\, g \in GL(V)^W \mid Gr(g) \in Aut_{\mathbb{R}}(\mathcal{S}) \,\}$$

acts transitively on $\mathcal{M}$. Moreover, given an element $F \in \mathcal{M}$, there exists a neighborhood U of $1 \in G_{\mathbb{C}}$ such that

$$g_{\mathbb{C}} \in U \implies g_{\mathbb{C}}.F \in \mathcal{M} \tag{3.10}$$

Proof. The proof of the fact that G acts transitively on $\mathcal{M}$ follows mutatis mutandis from the proof of Lemma (3.8).

To verify equation (3.10), pick a point $F \in \mathcal{M}$ and let $Y = Y_{(F,W)}$ be the grading defined in Example (2.17). Then, as may be verified by direct computation:

$$Lie(G_{\mathbb{C}}) = Lie(G_{\mathbb{C}}^Y) \oplus Lie_{-1}$$

Moreover, the subspace

$$\mathcal{C} = \left(\bigoplus_{p>0} \mathfrak{g}^{p,-p}\right) \bigoplus \sqrt{-1}\, \big(Lie(G_{\mathbb{C}}^F) \cap Lie(G^Y)\big)$$

is a vector space complement to $Lie(G^Y)$ in $Lie(G_{\mathbb{C}}^Y)$. Consequently, over a sufficiently small neighborhood U_0 of zero in $Lie(G_{\mathbb{C}})$, every element

$$g_{\mathbb{C}} \in \exp(U_0)$$

will admit a unique decomposition

$$g_{\mathbb{C}} = g_{-1} g^Y g_{\mathbb{C}}^F$$

with $g_{-1} \in \exp(Lie_{-1})$, $g^Y \in \exp(Lie(G^Y))$ and $g_{\mathbb{C}}^F$ in $\exp(\mathcal{C}) \subset G_{\mathbb{C}}^F$. In particular,

$$g_{\mathbb{C}} \in \exp(U_0) \implies g_{\mathbb{C}}.F = g_{-1} g^Y g_{\mathbb{C}}^F.F = g_{-1} g^Y.F \in \mathcal{M}$$

Remark. $\exp(Lie_{-1})$ *is a subgroup of G.*

As in the pure case, the relationship between variations of graded-polarized mixed Hodge structures (VGPMHS) and the corresponding classifying spaces of graded-polarized mixed Hodge structures is as follows: Let $\mathcal{V} \to S$ be a VGPMHS. Then, choice of a base point $s_0 \in S$ determines a monodromy representation

$$\rho : \pi_1(S, s_0) \to \Gamma \tag{3.11}$$

and a (locally-liftable) holomorphic, horizontal map

$$\phi : S \to \mathcal{M}/\Gamma \tag{3.12}$$

via parallel translation of the data of $\mathcal{V}$ to the reference fiber $V = \mathcal{V}_{s_0}$.

Conversely, given the monodromy representation (3.11) and the period map (3.12), it is possible to reconstruct the original variation $\mathcal{V}$ (up to isomorphism) by reversing the preceding construction.

To close this section, we shall now study the relationship between $Lie(G_{\mathbb{C}})$ and $\mathcal{M}$ in a bit more detail:

Theorem 3.13 *At each point $F \in \mathcal{M}$, the map*

$$u \in q_F \mapsto \exp(u).F \in \check{\mathcal{M}} \tag{3.14}$$

restricts to a biholomorphism from a neighborhood of zero in the nilpotent subalgebra

$$q_F = \bigoplus_{r<0, r+s\leq 0} \mathfrak{g}^{r,s} \subseteq Lie(G_{\mathbb{C}}) \tag{3.15}$$

to a neighborhood of $F \in \mathcal{M}$.

Proof. Because $G_{\mathbb{C}}$ acts transitively on $\check{\mathcal{M}}$ and $\mathcal{M}$ is an open subset of $\check{\mathcal{M}}$, it suffices to check that q_F is a vector space complement of $Lie(G_{\mathbb{C}}^F)$ in $Lie(G_{\mathbb{C}})$. To verify this last assertion, observe that

$$Lie(G_{\mathbb{C}}^F) = \bigoplus_{r\geq 0, r+s\leq 0} \mathfrak{g}^{r,s}$$

and hence

$$Lie(G_{\mathbb{C}}^F) \oplus q_F = \bigoplus_{r+s\leq 0} \mathfrak{g}^{r,s} = Lie(G_{\mathbb{C}})$$

Corollary 3.16 *The map $q_F \to T_F(\mathcal{M})$ which sends the endomorphism $u \in q_F$ to the derivation*

$$u(\zeta) = \frac{d}{dt}\zeta(\exp(tu).F)|_{t=0}$$

is a $\mathbb{C}$-linear isomorphism. In particular, the derivative Φ_* of a holomorphic, horizontal map $\Phi : S \to \mathcal{M}$ takes values in the horizontal subbundle

$$T_F^{horiz}(\mathcal{M}) = \bigoplus_{k\leq 1} \mathfrak{g}^{-1,k}_{(F,W)}, \qquad F \in \mathcal{M}$$

Remark. *The methods developed above may also be used to show that the real Lie group*

$$G_{\mathbb{R}} = \{\, g \in GL(V_{\mathbb{R}})^W \mid Gr(g) \in Aut_{\mathbb{R}}(\mathcal{S})\,\} \tag{3.17}$$

acts transitively on the C^∞ *submanifold* $\mathcal{M}_{\mathbb{R}} \subseteq \mathcal{M}$ *consisting of those filtrations* $F \in \mathcal{M}$ *for which the corresponding mixed Hodge structure* (F, W) *is split over* $\mathbb{R}$, *i.e.*

$$\overline{I^{p,q}_{(F,W)}} = I^{q,p}_{(F,W)}$$

Moreover, as may be easily checked by direct computation,

$$g \in G_{\mathbb{R}},\ F \in \mathcal{M} \implies I^{p,q}_{(g.F,W)} = g.I^{p,q}_{(F,W)} \tag{3.18}$$

In particular, the action of $G_{\mathbb{R}}$ *on* $\mathcal{M}$ *preserves the submanifold* $\mathcal{M}_{\mathbb{R}}$.

4 Deligne-Hodge bundles

Let $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ be a finite dimensional complex vector space which is defined over $\mathbb{Q}$. Then, as discussed in §2, each choice of a mixed Hodge structure (F, W) on V determines a unique, functorial decomposition

$$V = \bigoplus_{p,q} I^{p,q}_{(F,W)} \tag{4.1}$$

with the following three properties:

(1) $F^p = \oplus_{a \geq p, b}\, I^{a,b}$.

(2) $W_k = \oplus_{a+b \leq k}\, I^{a,b}$.

(3) $\bar{I}^{p,q} = I^{q,p} \mod \Lambda^{q-1,p-1}(V)$.

Consequently, each classifying space of graded-polarized mixed Hodge $\mathcal{M}$ modeled on V supports a natural decomposition

$$E = \bigoplus_{p,q} \mathcal{I}^{p,q} \tag{4.2}$$

of the corresponding trivial bundle $E = V \times \mathcal{M}$ into a sum of C^∞ subbundles

$$\mathcal{I}^{p,q}_F = I^{p,q}_{(F,W)} \tag{4.3}$$

To understand the first order behavior of the decomposition (4.2) relative to the flat connection

$$\nabla : \mathcal{E}^0(E) \to \mathcal{E}^1(E)$$

defined by exterior differentiation, recall that as discussed at the end of §2, the Lie group

$$G_{\mathbb{R}} = \{\, g \in GL(V_{\mathbb{R}})^W \mid Gr(g) \in Aut_{\mathbb{R}}(\mathcal{S})\,\} \tag{4.4}$$

act transitively upon the set of "real points" $\mathcal{M}_{\mathbb{R}} \subseteq \mathcal{M}$. Moreover, by virtue of equation (3.18):

$$g \in G_{\mathbb{R}}, F \in \mathcal{M} \implies \mathcal{I}^{p,q}_{g.F} = g.I^{p,q}_{(F,W)}$$

Consequently, it is relatively easy to understand the behavior of the decomposition (4.2) along $\mathcal{M}_{\mathbb{R}}$.

Unfortunately however, the group $G_{\mathbb{R}}$ does not (in general) act transitively upon the entire classifying space $\mathcal{M}$. Therefore, to apply the methods of the preceding paragraph to study the local behavior of the decomposition (4.4) near a given point $F \in \mathcal{M}$, we must construct a C^{∞} decomposition of each element $g_{\mathbb{C}} \in G_{\mathbb{C}}$ into a product

$$g_{\mathbb{C}} = g_{\mathbb{R}}\tilde{g}g^F_{\mathbb{C}} \tag{4.5}$$

with the following two properties

- $g_{\mathbb{R}} \in G_{\mathbb{R}}$, $g \in G$, $g^F_{\mathbb{C}} \in G^F_{\mathbb{C}}$.
- $I^{p,q}_{(\tilde{g}.F,W)} = \tilde{g}.I^{p,q}_{(F,W)}$.

In fact, provided that we are only interested in the local behavior of the Deligne–Hodge bundles, it will suffice to construct (4.5) over a neighborhood of $1 \in G_{\mathbb{C}}$.

Now, as discussed in [13], there exists a large class of C^{∞} decompositions of the form (4.5) which are in some sense "natural". However, for the task at hand, the decomposition determined by the following theorem appears to be the most suitable:

Theorem 4.6 *Let F be a point of $\mathcal{M}$ and $\Lambda^{-1,-1}_{(F,W)}$ be the nilpotent subalgebra of Lie_{-1} defined by the rule*

$$\Lambda^{-1,-1}_{(F,W)} = \bigoplus_{r,s<0} \mathfrak{g}^{r,s}$$

Then,

$$g \in \exp(\Lambda^{-1,-1}_{(F,W)}) \implies \mathcal{I}^{p,q}_{g.F} = g.\mathcal{I}^{p,q}_F$$

Moreover, there exists a natural $\mathbb{R}$-vector subspace $\Phi_F \subset Lie(G_{\mathbb{C}})$ such that

$$Lie(G_{\mathbb{C}}) = Lie(G_{\mathbb{R}}) \oplus \sqrt{-1}\left(\Lambda^{-1,-1}_{(F,W)} \cap Lie(G_{\mathbb{R}})\right) \oplus \Phi_F \tag{4.7}$$

Corollary 4.8 *Let F be an element of $\mathcal{M}$. Then, there exists a neighborhood $\exp(U_0)$ about $1 \in G_{\mathbb{C}}$ such that each element $g_{\mathbb{C}}$ in $\exp(U_0)$ admits a unique, C^{∞} decomposition*

$$g_{\mathbb{C}} = g_{\mathbb{R}}e^{\lambda}e^{\phi} \tag{4.10}$$

such that

- *$g_{\mathbb{R}}$ is an element of $G_{\mathbb{R}}$.*

- e^{λ} *is an element of* $\exp(\sqrt{-1}\,\Lambda^{-1,-1}_{(F,W)} \cap Lie(G_{\mathbb{R}}))$.
- $\exp(\phi)$ *is an element of* $\exp(\Phi_F)$.

Moreover, along the subalgebra q_F*,*

$$\phi(u) = -\pi_+(\bar{u}) + \textit{(higher order terms in } u \textit{ and } \bar{u}\textit{)}$$

The proof of Theorem (4.6) and Corollary (4.8) will occupy the remainder of this section. In essence however, the proofs of these two results boil down to a series of relatively straightforward calculations.

Lemma 4.10 *Let* F *be a point of* $\mathcal{M}$*. Then, the corresponding subgroup*

$$\exp(\Lambda^{-1,-1}_{(F,W)}) \subseteq \exp(Lie_{-1})$$

is closed under conjugation.

Proof. As discussed in §2, $\bar{\mathfrak{g}}^{r,s} = \mathfrak{g}^{s,r} \mod \oplus_{a<s,b<r}\ \mathfrak{g}^{a,b}$. Consequently,

$$\begin{aligned} \overline{\Lambda^{-1,-1}_{(F,W)}} &= \overline{\bigoplus_{r,s<0} \mathfrak{g}^{r,s}} = \bigoplus_{r,s<0} \bar{\mathfrak{g}}^{r,s} = \bigoplus_{r,s<0} \left(\mathfrak{g}^{s,r} \mod \bigoplus_{a<s,b<r} \mathfrak{g}^{a,b}\right) \\ &= \bigoplus_{r,s<0} \mathfrak{g}^{s,r} = \Lambda^{-1,-1}_{(F,W)} \end{aligned}$$

Lemma 4.11 *Let* F *be a point of* $\mathcal{M}$*. Then,*

$$g \in \exp(\Lambda^{-1,-1}_{(F,W)}) \implies \mathcal{I}^{p,q}_{g.F} = g.\mathcal{I}^{p,q}_F$$

Proof. Let $\{J^{p,q}\}$ denote the bigrading of $(g.F, W)$ defined by the rule

$$J^{p,q} = g.I^{p,q}_{(F,W)}$$

and note that, by construction, each element $h \in \exp(\Lambda^{-1,-1}_{(F,W)})$ preserves the condition

$$v \in I^{q,p}_{(F,W)} \mod \Lambda^{q-1,p-1}(V)$$

In particular, because the subgroup $\exp(\Lambda^{-1,-1}_{(F,W)})$ is closed under conjugation:

$$\begin{aligned} \bar{J}^{p,q} &= \bar{g}.\bar{I}^{p,q} = \bar{g}.\left(I^{q,p} \mod \Lambda^{q-1,p-1}(V)\right) \\ &= g\left(g^{-1}\bar{g}\right).\left(I^{q,p} \mod \Lambda^{q-1,p-1}(V)\right) \\ &= g.\left(I^{q,p} \mod \Lambda^{q-1,p-1}(V)\right) \\ &= J^{q,p} \mod \bigoplus_{a<q,b<p} J^{a,b} \end{aligned}$$

Thus, by uniqueness, $J^{p,q} = I^{p,q}_{(g.F,W)}$.

Armed with these two preliminary lemmata, we are now ready to complete the proofs of Theorem (4.6) and Corollary (4.8):

Proof. [Theorem (4.6)] Observe first that each point $F \in \mathcal{M}$ determines a vector space decomposition of $Lie(G_{\mathbb{C}})$ into direct sum of subalgebras

$$\begin{array}{ll} \eta_+ = \bigoplus_{r\geq 0,\, s<0} \mathfrak{g}^{r,s}, & \eta_0 = \mathfrak{g}^{0,0} \\ \eta_- = \bigoplus_{s\geq 0,\, r<0} \mathfrak{g}^{r,s}, & \Lambda^{-1,-1} = \Lambda^{-1,-1}_{(F,W)} \end{array} \tag{4.12}$$

with the following properties:

$$\begin{array}{ll} q_F = \eta_- \oplus \Lambda^{-1,-1}_F, & Lie(G^F_{\mathbb{C}}) = \eta_+ \oplus \eta_0 \\ \bar{\eta}_+ \subseteq \eta_- \oplus \Lambda^{-1,-1}, & \bar{\eta}_0 \subseteq \eta_0 \oplus \Lambda^{-1,-1} \\ \bar{\eta}_- \subseteq \eta_+ \oplus \Lambda^{-1,-1}, & \bar{\Lambda}^{-1,-1} = \Lambda^{-1,-1} \end{array} \tag{4.13}$$

Next, let π_+, π_0, π_- and π_Λ denote projection from $Lie(G_{\mathbb{C}})$ to the corresponding subalgebras η_+, η_0, η_- and $\Lambda^{-1,-1}$ listed in equation (4.12), and define

$$\Phi_F = \eta_+ \oplus \{\, x \in \eta_0 \mid \pi_0(\bar{x}) = -\pi_0(x) \,\} \tag{4.14}$$

Finally, observe that since equation (4.7) is a linear condition, it will suffice to check its validity on each of the subalgebras appearing in equation (4.12). Direct computation shows that:

$$\begin{array}{lll} x & \in \eta_+ & \Longrightarrow\ x = [0] \oplus [0] \oplus [x] \\ x & \in \eta_0 & \Longrightarrow\ x = [\mathrm{Re}(x)] \oplus [\pi_\Lambda(\mathrm{Im}(x))] \oplus [\pi_0(\mathrm{Im}(x))] \\ x & \in \eta_- & \Longrightarrow\ x = [\mathrm{Re}(2x - \pi_\Lambda(\bar{x}))] \oplus [-\mathrm{Im}(\pi_\Lambda(\bar{x}))] \oplus [-\pi_+(\bar{x})] \\ x & \in \Lambda^{-1,-1} & \Longrightarrow\ x = [\mathrm{Re}(x)] \oplus [\mathrm{Im}(x)] \oplus [0] \end{array}$$

where as usual,

$$\alpha \in Lie(G_{\mathbb{C}}) \implies \left\{ \begin{array}{l} \mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \bar{\alpha}) \\ \mathrm{Im}(\alpha) = \frac{1}{2}(\alpha - \bar{\alpha}) \end{array} \right.$$

For example, $x \in \eta_- \implies \bar{x} = \pi_\Lambda(\bar{x}) + \pi_+(\bar{x})$ and hence

$$\begin{array}{l} \mathrm{Re}(2x - \pi_\Lambda(\bar{x})) - \mathrm{Im}(\pi_\Lambda(\bar{x}) - \pi_+(\bar{x})) \\ \quad = \quad x + \bar{x} - \dfrac{1}{2}(\pi_\Lambda(\bar{x}) + \overline{\pi_\Lambda(\bar{x})}) - \dfrac{1}{2}(\pi_\Lambda(\bar{x}) - \overline{\pi_\Lambda(\bar{x})}) - \pi_+(\bar{x}) \\ \quad = \quad x + \bar{x} - \pi_\Lambda(\bar{x}) - \pi_+(\bar{x}) = x \end{array}$$

Proof. [Corollary (4.8)] Let u be an element of $U_0 \cap q_F$

$$e^u = g_{\mathbb{R}}(u) e^{\lambda(u)} e^{\phi(u)}$$

denote the decomposition of the element $e^u \in \exp(U_0)$ defined by equation (4.9), and define $\gamma(u) \in Lie(G_{\mathbb{R}})$ by the rule

$$g_{\mathbb{R}}(u) = e^{\gamma(u)}$$

Then, applying the Campbell–Baker–Hausdorff formula, one finds that

$$e^u = e^{\gamma(u)} e^{\lambda(u)} e^{\phi(u)} = e^{\gamma(u)+\lambda(u)+\phi(u)+(\text{higher order brackets})} \tag{4.15}$$

To use equation (4.15) to determine the first order behavior of $\phi(u)$, one simply inserts the first order Taylor series expansions

$$\gamma(u) = \gamma_1(u) + O^2(u), \quad \lambda(u) = \lambda_1(u) + O^2(u), \quad \phi(u) = \phi_1(u) + O^2(u)$$

into expression (4.15), thereby obtaining the equation

$$e^u = e^{\gamma_1(u)+\lambda_1(u)+\phi_1(u)+O^2(u)} \tag{4.16}$$

Comparing the linear terms on each side of equation (4.16), it therefore follows that

$$u = \gamma_1(u) + \lambda_1(u) + \phi_1(u) \tag{4.17}$$

Applying our previous formulae to equation (4.17), and remembering that u is an element of q_F, we obtain the desired result. Namely:

$$\phi_1(u) = -\pi_+(\bar{u})$$

Remark. *Throughout this paper I shall use the symbol $O^2(x)$ to denote an error term of order 2 depending (in principle) upon both x and $\bar{x}$.*

5 Higgs Fields

In this section, we prove the main theorem of this paper, namely:

Theorem 5.1 *Let $V \to S$ be a variation of graded-polarized mixed Hodge structure, and $\{\mathcal{U}^p\}$ denote the collection of C^∞ subbundles of $\mathcal{V}$ defined by the rule:*

$$\mathcal{U}^p_s = \bigoplus_q I^{p,q}_{(\mathcal{F}_s,\mathcal{W}_s)} \tag{5.2}$$

Then, relative to the Gauss–Manin connection ∇, the direct sum decomposition

$$\mathcal{V} = \bigoplus_p \mathcal{U}^p$$

defines a (unpolarized) complex variation of Hodge structure.

In particular, by virtue of our discussions in §1, the preceding result has the following immediate corollary:

Corollary 5.3 *Every variation of graded-polarized mixed Hodge structure V supports a natural Higgs bundle structure $(\mathcal{V},\bar{\partial}+\theta)$. Moreover, because this Higgs bundle structure arises from a complex variation of Hodge structure, it is automatically a fixed point of the $\mathbb{C}^*$ action:*

$$(\mathcal{V},\bar{\partial}+\theta)\mapsto(\mathcal{V},\bar{\partial}+\lambda\theta)$$

The formal proof of Theorem (5.1) presented below depends upon a couple of technical computations [namely: Lemma (5.11)]. The gist of the proof however is relative simple, and may be outlined as follows: The triple $(\mathcal{V},\{\mathcal{U}^p\},\nabla)$ defines a complex variation of Hodge structure if and only if differentiation induces a map

$$\nabla:\mathcal{E}^0(\mathcal{U}^p)\to\mathcal{E}^{0,1}(\mathcal{U}^{p+1})\oplus\mathcal{E}^1(\mathcal{U}^p)\oplus\mathcal{E}^{1,0}(\mathcal{U}^{p-1}) \tag{5.4}$$

Thus, in order to prove Theorem (5.1), it will suffice to compute the derivative of an arbitrary C^∞ local section σ of $\mathcal{U}^p$ at a given point $\underline{s}\in S$.

In particular, because the value of $\nabla\sigma$ at $\underline{s}$ is completely determined by the local behavior of $\mathcal{V}$, we may assume that our variation is defined over the polydisk

$$\Delta^n=\{(s_1,\dots,s_n)\in\mathbb{C}^n\mid |s_j|<1\quad j=1,\dots,n\}$$

via a holomorphic, horizontal map

$$F(s):\Delta^n\to\mathcal{M} \tag{5.5}$$

We may also assume that our given point $\underline{s}\in S$ corresponds to the point $0=(0,\dots,0)\in\Delta^n$.

In particular, by virtue of Theorem (3.13), there exists a neighborhood O about $0\in\Delta^n$ over which the period map (5.5) admits a unique representation

$$F(s)=e^{\Gamma(s)}.F(0) \tag{5.6}$$

relative to a holomorphic function $\Gamma(s)$ which take values in $q_{F(0)}$ and vanishes at zero.

To compute $\nabla\sigma$, observe that in light of Corollary (4.8), we may decompose the function $e^{\Gamma(s)}$ into a product of three factors

$$e^{\Gamma(s)}=g_{\mathbb{R}}(s)e^{\lambda(s)}e^{\phi(s)} \tag{5.7}$$

such that

- $g_{\mathbb{R}}(s)$ takes values in $G_{\mathbb{R}}$.
- $e^{\lambda(s)}$ takes values in $\exp(\sqrt{-1}\Lambda^{-1,-1}_{(F(0),W)}\cap Lie(G_{\mathbb{R}}))$.
- $e^{\phi(s)}$ takes values in $\exp(\Phi_{F(0)})$.

Consequently, the section $\sigma(s)$ may be written in the form

$$\sigma(s) = g_{\mathbb{R}}(s)e^{\lambda(s)}.\tilde{\sigma}(s) \tag{5.8}$$

relative to a smooth function $\tilde{\sigma}(s)$ which takes values in the fixed vector subspace

$$\mathcal{U}^p_{F(0)} = \bigoplus_q I^{p,q}_{(F(0),W)} \tag{5.9}$$

By Leibniz's rule:

$$\nabla\sigma(s) = \left(\nabla g_{\mathbb{R}}(s)e^{\lambda(s)}\right).\tilde{\sigma} + \left(g_{\mathbb{R}}(s)e^{\lambda(s)}\right).\nabla\tilde{\sigma} \tag{5.10}$$

Thus, to complete the proof of Theorem (5.1), it will suffice to compute the value of equation (5.10) at $s = 0$. To this end, we shall employ the following lemma:

Lemma 5.11 *Let F be a point of $\mathcal{M}$ and u be an element of $T^{horiz}_F(\mathcal{M})$. Then, relative to isomorphism $q_F \cong T_F(\mathcal{M})$ determined by Corollary* (3.16)*,*

$$\pi_+(\bar{u}) \in \mathfrak{g}^{1,-1} \oplus \left(\bigoplus_{k\leq -1} \mathfrak{g}^{0,k}\right)$$

In particular,

$$\pi_+(\bar{u}) : \mathcal{U}^p_{F(0)} \to \mathcal{U}^{p+1}_{F(0)} \oplus \mathcal{U}^p_{F(0)}$$

Proof. Given element $\alpha \in Lie(G_{\mathbb{C}})$ and a point $F \in \mathcal{M}$, let

$$\alpha = \sum_{r+s\leq 0} \alpha^{r,s}, \qquad \alpha^{r,s} \in \mathfrak{g}^{r,s}$$

denote the decomposition of α according to the induced bigrading

$$Lie(G_{\mathbb{C}}) = \bigoplus_{r+s\leq 0} \mathfrak{g}^{r,s}$$

Then, by Corollary (3.16),

$$u \in T^{horiz}_F(\mathcal{M}) \implies u = \sum_{k\leq 1} u^{-1,k}$$

Thus,

$$u = u^{-1,1} + u^{-1,0} + \sum_{k\leq -1} u^{-1,k} \tag{5.12}$$

In particular, because the rightmost term $\sum_{k\leq -1} u^{-1,k}$ of equation (5.12) is an element of $\Lambda^{-1,-1}_{(F,W)}$ and $\Lambda^{-1,-1}_{(F,W)} = \overline{\Lambda^{-1,-1}_{(F,W)}}$, we have:

$$\bar{u} = \overline{u^{-1,1}} + \overline{u^{-1,0}} \mod \Lambda^{-1,-1}_{(F,W)}$$

To finish the proof, recall that

$$\bar{\mathfrak{g}}^{r,s} = \mathfrak{g}^{s,r} \mod \bigoplus_{a<s,b<r} \mathfrak{g}^{a,b}$$

Consequently,

$$\bar{u} \in \mathfrak{g}^{1,-1} \oplus \left(\sum_{k \leq -1} \mathfrak{g}^{0,k} \right) \mod \Lambda^{-1,-1}_{(F,W)} \tag{5.13}$$

Proof. [Theorem (5.1)] Let $\nabla = \nabla^{0,1} + \nabla^{1,0}$ denote the decomposition of the Gauss-Manin connection ∇ into its holomorphic and anti-holomorphic parts. Then, by virtue of equation (5.10), Lemma (5.11), and the fact that

$$\Gamma(0) = 0 \implies g_{\mathbb{R}}(0) = 1, \quad e^{\lambda(0)} = 1, \quad \tilde{\sigma}(0) = \sigma(0)$$

it will suffice to show that [relative to the isomorphism $T_{F(0)}(\mathcal{M}) \cong q_{F(0)} \subset Lie(G_{\mathbb{C}})$]:

$$\begin{aligned} \nabla^{0,1}\, g_{\mathbb{R}}(s)e^{\lambda(s)}\big|_{s=0} &\in \pi_+\left(\overline{T^{horiz}_{F(0)}(M)}\right) \otimes T^{0,1}_0(\Delta^n)^* \\ \nabla^{1,0}\, g_{\mathbb{R}}(s)e^{\lambda(s)}\big|_{s=0} &\in T^{horiz}_{F(0)}(\mathcal{M}) \otimes T^{1,0}_0(\Delta^n)^* \end{aligned} \tag{5.14}$$

To this end, let

$$\Gamma(s) = \xi_1 s_1 + \cdots + \xi_n s_n + O^2(s) \tag{5.15}$$

denote the first order Taylor series expansion of $\Gamma(s)$. Then, equation (5.6) together with the horizontality of $\mathcal{V}$ imply that

$$\xi_j \in T^{horiz}_{F(0)}(\mathcal{M}), \qquad j = 1, \ldots, n \tag{5.16}$$

Next, observe that by virtue of equation (5.7),

$$g_{\mathbb{R}}(s)e^{\lambda(s)} = e^{\Gamma(s)}e^{-\phi(s)} \tag{5.17}$$

Moreover, equations (5.15) and (5.16) together with Corollary (4.8), imply that

$$\phi(s) = -\sum_{j=1}^{n} \pi_+(\bar{\xi}_j)\bar{s}_j + O^2(s) \tag{5.18}$$

Thus, by the Campbell–Baker–Hausdorff formula,

$$e^{\Gamma(s)}e^{-\phi(s)} = \exp(\sum_{j=1}^{n} \xi_j s_j + \sum_{j=1}^{n} \pi_+(\bar{\xi}_j)\bar{s}_j + O^2(s)) \tag{5.19}$$

In particular,

$$\begin{aligned} \nabla^{0,1}\, g_{\mathbb{R}}(s)e^{\lambda(s)}\big|_{s=0} &= \textstyle\sum_{j=1}^n \pi_+(\bar{\xi}_j) \otimes d\bar{s}_j \\ \nabla^{1,0}\, g_{\mathbb{R}}(s)e^{\lambda(s)}\big|_{s=0} &= \textstyle\sum_{j=1}^n \xi_j \otimes ds_j \end{aligned} \tag{5.20}$$

Corollary 5.21 *The Higgs bundle structure $\bar{\partial} + \theta$ associated to a variation of graded-polarized mixed Hodge structure $\mathcal{V} \to S$ preserves the weight filtration $\mathcal{W}$.*

To state the next result, let $(E, \{\mathcal{U}^p\}, \nabla)$ be a complex variation of Hodge structure. Then, as discussed in §1, the corresponding Higgs bundle $(E, \bar{\partial} + \theta)$ is obtained by using horizontality condition

$$\nabla : \mathcal{E}^0(\mathcal{U}^p) \to \mathcal{E}^{0,1}(\mathcal{U}^{p+1}) \oplus \mathcal{E}^{0,1}(\mathcal{U}^p) \oplus \mathcal{E}^{1,0}(\mathcal{U}^p) \oplus \mathcal{E}^{1,0}(\mathcal{U}^{p-1})$$

to decompose ∇ into the sum of a pair of differential operators

$$\bar{\partial} : \mathcal{E}^0(\mathcal{U}^p) \to \mathcal{E}^{0,1}(\mathcal{U}^p), \qquad \partial : \mathcal{E}^0(\mathcal{U}^p) \to \mathcal{E}^{1,0}(\mathcal{U}^p)$$

and a pair of tensor fields

$$\tau \in Hom(\mathcal{U}^p, \mathcal{U}^{p+1}) \otimes \mathcal{E}^{0,1}, \qquad \theta \in Hom(\mathcal{U}^p, \mathcal{U}^{p-1}) \otimes \mathcal{E}^{1,0}$$

Remark. *During the remainder of this paper, we shall also use the following notation: Let α be an element of $Lie(G_{\mathbb{C}})$ and y be a grading of W. Then, α^Y will denote the component of α which is of weight zero relative to the semi-simple endomorphism $ad\,Y$.*

Lemma 5.22 *Let $\mathcal{V}$ be a variation of graded-polarized mixed Hodge structure with Deligne grading*

$$\mathcal{Y}(\sigma) = k\sigma \iff \sigma \in \mathcal{E}^0\left(\bigoplus_{p+q=k} \mathcal{I}^{p,q}\right)$$

and Gauss–Manin connection $\nabla = \tau + \bar{\partial} + \partial + \theta$, Then,

$$\tau = (\bar{\theta})^{\mathcal{Y}} \tag{5.23}$$

Corollary 5.24 *Let $\mathcal{V} \to S$ be a VGPMHS. Then, $\mathcal{U}^p$ is a holomorphic subbundle of $\mathcal{V}$ relative to the integrable complex structure $\nabla^{0,1}$ iff $\bar{\theta}^Y$ vanishes on $\mathcal{U}^p$. In particular, assuming the system of Hodge bundles is of the form*

$$\mathcal{U}^a \oplus \mathcal{U}^{a+1} \oplus \cdots \oplus \mathcal{U}^b \tag{5.25}$$

it follows that $\mathcal{U}^b$ is a holomorphic subbundle of $\mathcal{V}$ (relative to the Gauss–Manin connection ∇).

The remainder of this section is devoted to some applications of Theorem (5.1) to the study of unipotent variations of mixed Hodge structure. Applications of Theorem (5.1) to quantum cohomology and mirror symmetry will be discussed in §7 and §8.

Definition 5.26 *A variation $\mathcal{V}$ is unipotent if and only if the induced variations $\mathcal{F}Gr^{\mathcal{W}}$ are constant.*

Lemma 5.27 *If $\mathcal{V}$ is unipotent then*

$$\theta : \mathcal{W}_k \to \mathcal{W}_{k-1} \tag{5.28}$$

for each index k. In particular, $(\bar{\theta})^{\mathcal{Y}} = 0$.

Proof. The induced maps $\theta^Y : Gr_k^{\mathcal{W}} \to Gr_k^{\mathcal{W}}$ coincide with the Higgs field carried by the corresponding variation on $Gr_k^{\mathcal{W}}$. However, by unipotency, the induced variations on Gr^W are constant.

Corollary 5.29 *If $\mathcal{V}$ is a unipotent VGPMHS then*

(1) The complex structures $\bar{\partial}$ and $\nabla^{0,1}$ coincide.

(2) The connection $\bar{\partial} + \partial = \nabla - \theta$ is flat.

(3) The Higgs field θ is flat relative both to ∇ and $\bar{\partial} + \partial$.

Theorem 5.30 *A unipotent VGPMHS $\mathcal{V} \to S$ may be recovered from the following data:*

(1) The flat connection ∇ of $\mathcal{V}$.

(2) The associated Higgs field θ.

(3) A single fiber $\mathcal{L}_{s_0}$ of $\mathcal{V}$.

Proof. Since the weight filtration $\mathcal{W}$ and the bilinear forms $\{\mathcal{S}_k\}$ are flat, the key step is to recover the Hodge filtration $\mathcal{F}$ via the subbundles $\{\mathcal{U}^p\}$. However, by (5.29), $\mathcal{U}^p$ is parallel with respect to the flat connection $\nabla - \theta$.

Now, as observed by Deligne [6], the class of unipotent variations described above contains the following special subclass which appears to be of great importance in mirror symmetry:

Definition 5.31 *A variation $\mathcal{V} \to S$ is said to be Hodge–Tate provided the graded Hodge numbers $h^{p,q}$ vanish unless $p = q$.*

Lemma 5.32 *Let $\mathcal{V}$ be a variation of Hodge–Tate type. Then, the corresponding Higgs field θ assumes values $\mathfrak{g}^{-1,-1}$.*

Proof. In this case, the horizontal subspace

$$T_{F(0)}^{horiz}(\mathcal{M}) \subseteq T_{F(0)}(\mathcal{M})$$

appearing in the proof of Theorem (5.1) reduces to $\mathfrak{g}^{-1,-1}_{(F(0),W)}$.

Corollary 5.33 *If $\mathcal{V}$ is a Hodge–Tate variation then*

(1) The grading $\mathcal{Y}$ is flat relative to the connection $\bar{\partial} + \partial$.

(2) $\nabla\mathcal{Y} = 2\theta$.

6 Asymptotic Behavior

In this section we consider the asymptotic behavior of variations of graded-polarized mixed Hodge structure $\mathcal{V} \to \Delta^{*n}$ which are admissible in the sense of [21] and [14].

To this end, recall that given a flat vector bundle $E \to \Delta^{*n}$ with unipotent monodromy, there exists a canonical extension $E^c \to \Delta^n$ relative to which the flat connection ∇ of E has at worst a simple poles with nilpotent residues along the divisor $D = \Delta^n - \Delta^{*n}$.

More explicitly, given a choice of coordiantes Δ^n relative to which

$$D = \{\, p \in \Delta^n \mid s_1(p)\cdots s_n(p) = 0 \,\} \tag{6.1}$$

one may identify E^c with the locally free sheaf generated by the sections

$$\sigma^c = e^{\sum_j \frac{1}{2\pi i}(\log s_j)N_j}\sigma, \quad \sigma \text{ a flat, multivalued section of } E \tag{6.2}$$

where $T_j(s) : E_s \to E_s$ denotes the action of parallel translation along the counterclockwise path $\gamma_j(t) = (s_1, \dots, s_{j-1}, e^{2\pi i t}s_j, s_{j+1}, \dots, s_n)$. and

$$N_j = -\log T_j$$

In particular, given the choice of coordinates (6.1), we obtain a flat coonection ∇^c on E^c via the rule:

$$\nabla^c = \nabla - \frac{1}{2\pi i}\sum_{j=1}^{n} \frac{ds_j}{s_j} \otimes N_j,$$

Remark. *Our sign conventions regarding N_j follow [6].*

Suppose now that $\mathcal{V} \to \Delta^{*n}$ is a variation of graded-polarized mixed Hodge structure with unipotent monodromy. Then, by virtue (6.2), we obtain

(1) A choice of rational structure $V_{\mathbb{Q}}$ on the central fiber of $\mathcal{V}^c$.

Relative $V_{\mathbb{Q}}$, parallel translation to the central fiber of E^c under ∇^c then defines:

(2) A rational weight filtration W of V.

(3) Rational, non-degenerate bilinear forms $\mathcal{S}_k : Gr^W_k \otimes Gr^W_k \to \mathbb{C}$ of parity $(-1)^k$.

(4) An "untwisted" period map $\psi : \Delta^{*n} \to \check{\mathcal{M}}$ encoding the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$.

Following [21] and [14] let us henceforth assume that $\mathcal{V}$ is admissible, i.e.

(5) The map $\psi : \Delta^{*n} \to \check{\mathcal{M}}$ extends to Δ^n.

(6) The data (1)–(3) together with the limiting filtration

$$F_\infty := \lim_{p\to 0} \psi(p) \tag{6.3}$$

and the monodromy logarithms $N_1, \dots, N_n$ define an infinitesimal mixed-Hodge module (IMHM) in the sense of [14].

Then, by the work of Deligne and Kashiwara, the monodromy cone

$$\mathcal{C} = \{ \sum_{j=1}^{n} a_j N_j \mid a_j > 0 \} \tag{6.4}$$

of $\mathcal{V}$ enjoys the following properties:

- The relative weight filtration ${}^rW = {}^rW(N, W)$ is constant on $\mathcal{C}$.
- The pair $(F_\infty, {}^rW)$ is a mixed Hodge structure.
- Each element N_j is a $(-1,-1)$ morphism of $(F_\infty, {}^rW)$.

Consequently, the limiting mixed Hodge structure $(F_\infty, {}^rW)$ of $\mathcal{V}$ defines a canonical decomposition

$$Lie(G_{\mathbb{C}}) = \bigoplus_a \wp_a, \qquad \wp_a : U^p_\infty \to U^{p+a}_\infty \tag{6.5}$$

via the subspaces

$$U^p_\infty = \bigoplus_q I^{p,q}_{(F_\infty, {}^rW)} \tag{6.6}$$

In particular, since $F^p_\infty = F^{p+1}_\infty \oplus U^p_\infty$, the graded, nilpotent Lie algebra

$$q_\infty = \bigoplus_{a<0} \wp_a \tag{6.7}$$

is a vector space complement to $Lie(G^{F_\infty}_{\mathbb{C}})$ in $Lie(G_{\mathbb{C}})$. As a result, there exists a polydisk Δ^n_r of radius r over which

$$\psi(s) = e^{\Gamma(s)}.F_\infty \tag{6.8}$$

relative to a unique holomorphic function $\Gamma : \Delta^n_r \to q_\infty$ vanishing at zero.

Remark. *Since the original period map $F(z)$ of $\mathcal{V}$ may be written as $F(z) = e^{zN}e^{\Gamma(s)}.F_\infty$, equation (6.8) may be viewed as a type of "group theoretic" version of Schmid's Nilpotent Orbit Theorem for admissible VGPMHS.*

Theorem 6.9 *The function* Γ *defined by equation* (6.8) *satisfies the differential equation*

$$e^{-ad\,\Gamma}\,\Omega + e^{-\Gamma}\,\partial e^{\Gamma} \in \wp_{-1}, \quad \Omega = \frac{1}{2\pi i}\sum_{j=1}^{n}\frac{ds_j}{s_j}\otimes N_j \tag{6.10}$$

Proof. Let

$$U_r^n = \{\,(z_1,\dots,z_n)\in\mathbb{C}^n \mid (e^{2\pi i z_1},\dots,e^{2\pi i z_n})\in\Delta_r^{*n}\,\}$$

cover Δ_r^{*n} via $s_j = e^{2\pi i z_j}$. Then, the horizontality of the Hodge filtration

$$F(z_1,\dots,z_n) = e^{\sum_{j=1}^{n} z_j N_j}\,e^{\Gamma(s_1,\dots,s_n)}.F_\infty \tag{6.11}$$

implies that

$$\left[e^{\sum_{j=1}^{n} z_j N_j}e^{\Gamma}\right]^{-1}\partial\left[e^{\sum_{j=1}^{n} z_j N_j}e^{\Gamma}\right]\in\wp_{-1} \tag{6.12}$$

which unravels to yield (6.10).

Theorem 6.13 *Let* X_{-k} *denote the component of*

$$X = \log\left(e^{\sum_{j=1}^{n} z_j N_j}e^{\Gamma(s_1,\dots,s_n)}\right)$$

taking values in $\wp_{-k}$*. Then, the endomorphism valued 1-form*

$$\partial X_{-1} = \Omega + \partial\Gamma_{-1} \tag{6.14}$$

defines a Higgs field on the product bundle $V\times\Delta_r^{*n}\to\Delta_r^{*n}$*.*

Proof. With a little work, equation (6.13) can be recast as

$$\partial e^X = e^X\,\partial X_{-1}$$

Consequently,

$$e^{-X}\frac{\partial^2}{\partial z_i\partial z_j}e^X = \frac{\partial X_{-1}}{\partial z_i}\frac{\partial X_{-1}}{\partial z_j} + \frac{\partial^2 X_{-1}}{\partial z_i\partial z_j}$$

and hence

$$\partial X_{-1}\wedge\partial X_{-1} = 0$$

by equality of mixed partial derivatives.

Remark. *As a result of Theorem* (6.13)*,*

$$\tilde{\nabla}\sigma = d\sigma + \partial X_{-1}(\sigma) \tag{6.15}$$

defines a flat connection on the product bundle $V\times\Delta_r^{*n}\to\Delta_r^{*n}$*.*

Comparing Theorem (6.13) with Theorem (5.30), it is natural to ask when a holomorphic function $\Gamma_{-1} : \Delta^n_r \to \wp_{-1}$ defines a solution to (6.10).

Theorem 6.16 *There exists a bijective correspondence between solutions to* (6.10) *and holomorphic functions* $\Gamma_{-1} : \Delta^n_r \to \wp_{-1}$ *which satisfy the Higgs field condition* $\partial X_{-1} \wedge \partial X_{-1} = 0$ *and the initial value constraint* $\Gamma_{-1}(0) = 0$*.*

I shall sketch two proofs of this result. The first, inspired by [6], is to show that Γ may be recovered from the monodromy action of the flat connection (6.15). More precisely, the monodromy of (6.15) about the loop $\gamma_j(t) = (s_1, \dots, e^{2\pi i t} s_j, \dots, s_n)$, $0 \leq t \leq 1$ is given by

$$T_j = e^{-\Gamma} e^{-N_j} e^{\Gamma} \tag{6.17}$$

The second method of proof, used by Cattani and Peters in their study of variations of pure Hodge structure, is to write

$$e^{\Gamma} = 1 + E_{-1} + E_{-2} + \cdots + E_{-m}, \quad E_{-k} : \Delta^n_r \to \wp_{-k} \tag{6.18}$$

and show that starting from $E_{-1} = \Gamma_{-1}$ it is possible to construct each function E_{-k} inductively using (6.10).

Now, because the "untwisted"period map $\psi : \Delta^n \to \check{\mathcal{M}}$ depends upon the choice of coordinates $(s_1, \dots, s_n)$ via the connection ∇^c, in order to state the next two results we must fix a choice of holomorphic coordinates $(s_1, \dots, s_n)$ on Δ^n relative to which the divisor $D = \Delta^n - \Delta^{*n}$ assumes the form $D = \{\, p \in \Delta^n \mid s_1(p) \cdots s_n(p) = 0 \,\}$.

Theorem 6.19 *An admissible VGPMHS* $\mathcal{V} \to \Delta^{*n}$ *may be recovered from the following data:*

(1) The flat connection ∇ *of* $\mathcal{V}$*.*

(2) The Higgs field ∂X_{-1}*.*

(3) The limiting IMHM (6.3)*.*

Proof. Since the coordinates $(s_1, \dots, s_n)$ are known, we may recover the connection ∇^c via (1). Therefore, by using the data contained in (3), we may recover rational structure of $\mathcal{V}$, the weight filtration $\mathcal{W}$, and the bilinear forms $\mathcal{S}_k$. To recover the map $\psi : \Delta^n \to \check{\mathcal{M}}$, and hence the Hodge filtration $\mathcal{F}$, observe that by holomorphicity it will suffice to determine ψ over any open subset of Δ^n. However, by Theorem (6.16) we can recover ψ on an neighborhood of zero using (2) and (3).

Corollary 6.20 *The machinery of Theorem* (6.19) *establishes an equivalence of categories between admissible VGPMHS* $\mathcal{V} \to \Delta^{*n}$ *which are unipotent in the sense of Definition* (5.26) *and the corresponding data* (1)–(3)*.*

Proof. The key step in establishing an equivalence of categories is that we must not shrink the domain of the Higgs field ∂X_{-1}. Equivalently, the representation $\psi = e^{\Gamma}.F_{\infty}$ must hold over all of Δ^n. To prove this, observe that by unipotency

$$\psi(\Delta^n) \subseteq \exp(Lie_{-1}).F_{\infty}$$

Example 6.21 *Let $\mathcal{V} \to \Delta^{*n}$ be an admissible variation of Hodge–Tate type. Then, because the variation is Hodge–Tate, the relative weight filtration ${}^rW$ coincides with the original weight filtration W and*

$$q_{\infty} = \Lambda^{-1,-1}_{(F_{\infty},W)}$$

Moreover, upon selecting suitable branches of $\log s$, the subbundle $\mathcal{U}^p$ of $\mathcal{V}$ may be identified with

$$\mathcal{U}^p(s) = e^{\frac{1}{2\pi i}\sum_{j=1}^n \log s_j \otimes N_j}\, e^{\Gamma(s)}.U^p_{\infty}$$

Therefore, by Theorem (5.1)*:*

$$\begin{aligned} \theta &= [\partial(e^{\frac{1}{2\pi i}\sum_{j=1}^n \log s_j\otimes N_j}\, e^{\Gamma})][e^{\frac{1}{2\pi i}\sum_{j=1}^n \log s_j\otimes N_j}\, e^{\Gamma}]^{-1} \\ &= \Omega + Ad(e^{\frac{1}{2\pi i}\sum_{j=1}^n \log s_j\otimes N_j})((\partial e^{\Gamma})e^{-\Gamma}) \\ &= \Omega + \partial\Gamma_{-1} \mod \bigoplus_{a\leq -2} \wp_a = \partial X_{-1} \mod \bigoplus_{a\leq -2} \wp_a \end{aligned}$$

Thus, the horizontal component of θ relative to (F_{∞}, W) coincides with the Higgs field ∂X_{-1}.

Remark. *Because of the graded structure of q_{∞}, there exist universal Lie polynomials $P_2, P_3, \ldots$ such that*

$$\partial\Gamma_{-k} = P_k(\Gamma_{-1}, \ldots, \Gamma_{1-k}, \partial\Gamma_{-1}, \ldots, \partial\Gamma_{1-k}, \Omega)$$

whenever Γ is a solution to (6.10)*. In particular, as they shall be of use in* §7*, let us observe that the first two of the equations are given by*

$$\begin{aligned} \partial\Gamma_{-2} &= [\Gamma_{-1}, \Omega + \frac{1}{2}\partial\Gamma_{-1}] \\ \partial\Gamma_{-3} &= [\Gamma_{-2}, \Omega + \frac{1}{2}\partial\Gamma_{-1}] + \frac{1}{12}[\Gamma_{-1}, [\Gamma_{-1}, \partial\Gamma_{-1}]] \end{aligned} \tag{6.22}$$

7 A–model

The relationship between the Higgs fields constructed in §5–6 and the quantum cohomology of Calabi–Yau threefolds may be roughly summarized as follows:

- The A–model correlation functions of a Calabi–Yau threefold X are completely determined by the Higgs field ∂X_{-1} of the corresponding A–model variation of Hodge structure described in [4].
- The B–model correlation functions of the corresponding mirror family X° are completely determined by the Higgs field θ of the corresponding Hodge–Tate variation described in [6].

In this section, we shall treat the A–model side of the story from the standpoint of the Higgs field ∂X_{-1}. The details of the B–model side will be treated in §8.

As a prelude to our discussion of the A–model variation of Hodge structure and its relationship to quantum cohomology, we need to first recall some standard terminology: Let X be a smooth Calabi–Yau threefold. Then, the corresponding Kähler cone $K(X)$ is just the set:

$$K(X) = \{\, \omega \in H^2(X, \mathbb{R}) \mid \omega \text{ is Kähler} \,\} \tag{7.1}$$

Likewise, complexified Kähler space $K_{\mathbb{C}}(X)$ is just the quotient of the set

$$\{\, \omega \in H^2(X, \mathbb{C}) \mid \mathrm{Im}(\omega) \in K(X) \,\}$$

by the torsion free part of $H^2(X, \mathbb{Z})$.

Definition 7.2 *A simplicial cone $\sigma \subset H^2(X, \mathbb{R})$ is said to be a large radius limit point of $K_{\mathbb{C}}(X)$ provided that it is of maximal dimension and satisfies $Int(\sigma) \subseteq K(X)$.*

To construct a nice partial compactification of $K_{\mathbb{C}}(X)$ about a given large radius limit point σ, let us suppose σ to be generated by a collection of basis vectors

$$T_1, \ldots, T_n \in (H^2(X, \mathbb{Z})/\text{torsion}) \cap \overline{K(X)} \tag{7.3}$$

Now, by virtue of (7.3)

$$Int(\sigma) = \{\, a_1 T_1 + \cdots + a_n T_n \mid a_1, \ldots, a_n > 0 \,\}$$

and hence the product of upper half-planes

$$U_\sigma^n = \{\, u_1 T_1 + \cdots + u_n T_n \mid \mathrm{Im}(u_1), \ldots, \mathrm{Im}(u_n) > 0 \,\}$$

projects onto a neighborhood of infinity D_σ of $K_{\mathbb{C}}(X)$ via the quotient map

$$\vec{u} \in U_\sigma^n \mapsto [\vec{u}] \in K_{\mathbb{C}}(X) \tag{7.4}$$

Consequently, we may use the quotient map (7.4) to obtain a completion

$$D_\sigma \xhookrightarrow{i} \overline{D}_\sigma$$

via the standard covering map

$$(u_1, \ldots, u_n) \in U_\sigma^n \mapsto (e^{2\pi i u_1}, \ldots, e^{2\pi i u_n}) \in \Delta^{*n}$$

and the inclusion $\Delta^{*n} \overset{i}{\hookrightarrow} \Delta^n$.

In particular, upon setting $q_j = e^{2\pi i u_j}$, we obtain a natural system of coordinates

$$(q_1, \ldots, q_n) : D_\sigma \to \Delta^{*n}$$

relative to which the completion $D_\sigma \overset{i}{\hookrightarrow} \overline{D}_\sigma$ becomes the standard embedding $\Delta^{*n} \overset{i}{\hookrightarrow} \Delta^n$.

In light of these observations, we shall henceforth assume that our large radius limit point σ comes equipped with a choice of basis $T_1, \ldots, T_n$ of $H^2(X, \mathbb{Z})$/torsion relative to which (7.3) holds.

The next ingredient the we must assemble in order to discuss the A–model connection is the small quantum product $a * b$.

Lemma 7.5 *Let σ be a large radius limit point of $K_{\mathbb{C}}(X)$. Then, relative to the coordinates*

$$\omega(u_1, \ldots, u_n) = \sum_{j=1}^{n} u_j T_j$$

defined on U_σ^n by the basis (7.3)*, the Gromov–Witten potential Φ of X assumes the form*

$$\Phi(u_1, \ldots, u_n) = \left(\frac{1}{6} \int_X \omega^3 \right) + \Phi_{hol}(q_1, \ldots, q_n)$$

with respect to the formal q-power series:

$$\Phi_{hol}(q_1, \ldots, q_n) = \frac{1}{(2\pi i)^3} \sum_{\beta \in H^2(X,\mathbb{Z})-\{0\}} \langle I_{0,0,\beta} \rangle \, e^{2\pi i \int_\beta \omega} \tag{7.6}$$

Corollary 7.7 *Let σ be a large radius limit point of $K_{\mathbb{C}}(X)$ and*

$$^\vee : H^k(X, \mathbb{C}) \to H^{6-k}(X, \mathbb{C})$$

denote the map induced by Poincaré duality. Then,

$$T_j * T_k = \sum_\ell \frac{\partial^3 \Phi}{\partial u_j \partial u_k \partial u_\ell} T_\ell^\vee$$

Remark. *The formal q-power series* (7.6) *is expected to converge for all values of q sufficiently small. Therefore, in order to simplify our exposition, for the remainder of this section we shall assume the sum* (7.6) *to converge over all of $\overline{D}_\sigma$.*

To construct the A–model variation of Hodge structure, let

$$H = \bigoplus_p H^{p,p}(X)$$

and $\mathcal{V}$ denote the product bundle $H \times D_\sigma$ endowed with the corresponding Dubrovin connection:

$$\nabla = d + A, \qquad A_{\frac{\partial}{\partial u_j}}\alpha = T_j * \alpha \tag{7.8}$$

Then, as may be easily checked by direct computation, the associativity of the small quantum product $*$ implies that ∇ is flat.

Thus, in order to finish the construction of the A–model variation, it remains for us to describe:

(1) The Hodge filtration $\mathcal{F}$.

(2) The polarization Q.

(3) The integral structure $\mathcal{V}_{\mathbb{Z}}$.

To endow $\mathcal{V}$ with an integral structure which is flat with respect to ∇, let $\mathcal{V}^c = H \times \overline{D}_\sigma$ denote the canonical extension of $\mathcal{V}$ and

$$\nabla^c = \nabla - \frac{1}{2\pi i}\sum_{j=1}^{n} \frac{dq_j}{q_j} \otimes N_j, \qquad N_j = -\log(T_j)$$

denote the corresponding connection of $\mathcal{V}^c$ defined by the choice of coordinates $q_1 = e^{2\pi i u_1}, \dots, q_n = e^{2\pi i u_n}$. Then, in keeping with our discussions in §6, it is natural to define it is natural to define

$$\sigma = e^{-\sum \frac{1}{2\pi i}\log(q_j)N_j}\sigma^c$$

to be a flat (multivalued) section of $\mathcal{V}_{\mathbb{Z}}$ if and only if σ^c is a ∇^c-flat section of $\mathcal{V}^c$ such that $\sigma^c(0) \in H^*(X,\mathbb{Z})$.

To polarize $\mathcal{V}$, we pair $\alpha \in H^{p,p}(X)$ and $\beta \in H^{3-p,3-p}(X)$ by the rule:

$$Q(\alpha,\beta) = (-1)^p \int_X \alpha \wedge \beta$$

Likewise, following [4], we define:

$$\mathcal{F}^p = \bigoplus_{a \le 3-p} H^{a,a}(X,\mathbb{C})$$

Theorem 7.9 *[4] The data $(\mathcal{V}_{\mathbb{Z}}, \mathcal{F}, Q)$ defines a variation of pure, polarized Hodge structure of weight* 3 *over a neighborhood of zero in D_σ for which the action of the monodromy logarithm N_j on the central fiber of $\mathcal{V}^c$ coincides with*

cup product with T_j. Thus, by virtue of the Hard Lefschetz Theorem, the weight filtration of the monodromy cone

$$\mathcal{C} = \{\, a_1 N_1 + \cdots + a_n N_n \mid a_1, \ldots, a_n > 0 \,\}$$

is given by the rule:

$$W_k = \bigoplus_{2a \geq 6-k} H^{a,a}(X)$$

Moreover, upon letting $^\vee : H^k(X,\mathbb{C}) \to H^{6-k}(X,\mathbb{C})$ denote the map induced by Poincaré duality and $T_0 = 1 \in H^0(X,\mathbb{C})$, the following sections may be shown to define a ∇^c-flat framing of $\mathcal{V}^c$:

$$\begin{aligned} \sigma_j &= T_j - \sum_\ell \frac{\partial^2 \Phi_{hol}}{\partial u_j \partial u_\ell}\, T_\ell^\vee + \frac{\partial \Phi_{hol}}{\partial u_j}\, T_0^\vee, \qquad && \sigma^j = T_j^\vee \\ \sigma_0 &= T_0 - \sum_\ell \frac{\partial \Phi_{hol}}{\partial u_\ell}\, T_\ell^\vee + 2\Phi_{hol}\, T_0^\vee, && \sigma^0 = T_0^\vee \end{aligned} \tag{7.10}$$

Armed with these preliminaries, we can now rewrite the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ in terms of the ∇^c-flat frame (7.10) and thus obtain the "untwisted" period map $\psi : \Delta^n \to \check{\mathcal{D}}$ described in §6. Upon carrying out these computations, one finds that

$$\begin{aligned} \mathcal{F}^3 &= \operatorname{span}_{\mathbb{C}}(\sigma_0 + \sum_\ell \frac{\partial \Phi_{hol}}{\partial u_\ell}\, \sigma^\ell - 2\Phi_{hol}\, \sigma^0) \\ \mathcal{F}^2 &= \mathcal{F}^3 \oplus \operatorname{span}_{\mathbb{C}}(\sigma_j + \sum_\ell \frac{\partial^2 \Phi_{hol}}{\partial u_j \partial u_\ell}\, \sigma^\ell - \frac{\partial \Phi_{hol}}{\partial u_j} \sigma^0) \\ \mathcal{F}^1 &= \mathcal{F}^2 \oplus \operatorname{span}_{\mathbb{C}}(\sigma^j), \qquad \mathcal{F}^0 = \mathcal{F}^1 \oplus \operatorname{span}_{\mathbb{C}}(\sigma^0) \end{aligned} \tag{7.11}$$

where the index j ranges from 1 to n.

Now, as may be easily checked, relative to definitions of $\mathcal{F}$ and $\mathcal{W}$ given above:

$$I^{p,p}_{(F_\infty, W)} = H^{3-p,3-p}(X,\mathbb{C}) \tag{7.12}$$

Therefore, by virtue of equation (7.11),

$$\psi(q_1, \ldots, q_n) = e^{\Gamma(q_1,\ldots,q_n)}.F_\infty$$

with $\Gamma = \Gamma_{-1} + \Gamma_{-2} + \Gamma_{-3}$, and

$$\begin{aligned} \Gamma_{-1}(T_k) &= \sum_{\ell=1}^{n} \frac{\partial^2 \Phi_{hol}}{\partial u_k \partial u_\ell}\, T_\ell^\vee \qquad && \Gamma_{-2}(T_k) = -\frac{\partial \Phi_{hol}}{\partial u_k}\, T_0^\vee \\ \Gamma_{-2}(T_0) &= \sum_\ell \frac{\partial \Phi_{hol}}{\partial u_\ell}\, T_\ell^\vee && \Gamma_{-3}(T_0) = -2\Phi_{hol}\, T_0^\vee \end{aligned} \tag{7.13}$$

Theorem 7.14 *Let σ be a large radius limit point of $K_{\mathbb{C}}(X)$ and*

$$\partial X_{-1} = \sum_{j=1}^{n} \left(N_j + \frac{\partial \Gamma_{-1}}{\partial u_j} \right) \otimes du_j$$

be the Higgs field of the corresponding A–model variation $\mathcal{V}$. Then, for any element $\alpha \in H$:

$$\partial X_{-1} \left(\frac{\partial}{\partial u_j} \right) \alpha = T_j * \alpha$$

Proof. Since N_j acts as cup product by T_j and $T_j * \alpha = T_j \cup \alpha$ provided α has no component in $H^2(X, \mathbb{C})$, it will suffice to establish the claim for $\alpha \in H^2(X, \mathbb{C})$. However, by equation (7.13),

$$\partial X_{-1} \left(\frac{\partial}{\partial u_j} \right) T_k = T_j \wedge T_k + \sum_{\ell} \frac{\partial^3 \Phi_{hol}}{\partial u_j \partial u_k \partial u_\ell} T_\ell^\vee = T_j * T_k$$

Definition 7.15 *Let M be a smooth projective variety, $e_0, \dots, e_m$ be a basis for the rational cohomology of M each term of which is homogeneous with respect to the degree map*

$$deg : H^*(M, \mathbb{Q}) \to \mathbb{Z}$$

and $t_0, \dots, t_m$ be the corresponding supercommuting variables defined by the rule:

$$t_j t_k = (-1)^{deg(e_j) deg(e_k)} t_k t_j$$

Then, a formal power series $\Phi(t_0, \dots, t_m)$ is said to be solution of the WDVV (Witten–Dijgraaf–Verlinde–Verlinde) equation provided the formal product

$$e_j * e_k = \sum_{\ell=0}^{m} \left(\frac{\partial^3 \Phi}{\partial t_j \partial t_k \partial t_\ell} \right) e_\ell^\vee \tag{7.16}$$

turns $H^*(M, \mathbb{C})$ into a supercommutative ring with identity element $1 \in H^0(X, \mathbb{C})$.

Example 7.17 *The standard cup product of M defines a natural solution to the WDVV equations via the potential function*

$$\Phi_0(t_0, \dots, t_m) = \int_M \left(\sum_{j=0}^{m} t_j e_j \right)^3$$

Another example of a solution to the WDVV equation is provided by the small quantum product. To relate these two solutions, observe that in the case of a Calabi–Yau threefold X, the small quantum product coincides with the usual cup product except on elements of $H^2(X, \mathbb{C})$. Moreover, upon selecting

a large radius limit point σ of $K_{\mathbb{C}}(X)$ and setting $e_j = T_j$ for $j = 1, \dots, n$, it follows by Corollary (7.7) that

$$\Phi(t_0, \dots, t_m) = \Phi_0(t_0, \dots, t_m) + \Phi_{hol}(q_1, \dots, q_n) \tag{7.18}$$

is a potential function for the small quantum product. In addition, because

$$\Phi_{hol}(0, \dots, 0) = 0 \tag{7.19}$$

the small quantum product is in fact a deformation of the standard potential function Φ_0.

Theorem 7.20 *Let σ be a large radius limit point of the Calabi–Yau threefold X. Then, there exists a bijective correspondence between germs of holomorphic solutions of the WDVV equations which satisfy conditions* (7.18)–(7.19) *and the germs of variations of pure, polarized Hodge structures of weight* 3 *on D_σ which are governed by equation* (7.13).

Proof. Given a variation of Hodge structure $\mathcal{V}$ governed by equation (7.13), one defines the corresponding potential Φ by the rule:

$$\Phi(t_0, \dots, t_m) = \Phi_0(t_0, \dots, t_m) + \Phi_{hol}(q_1, \dots, q_n) \tag{7.21}$$

where

$$\Gamma_{-3}(T_0) = -2\Phi_{hol}\, T_0^\vee$$

Now, in order to verify that equation (7.21) defines a solution of the WDVV equation, observe that because X is a Calabi–Yau threefold, it will suffice to check that

$$T_a * (T_b * T_c) = (T_a * T_b) * T_c$$

for all indices a, b and c.

To this end, note that by virtue of our assumption that $e_j = T_j$ for $j = 1, \dots, n$ and equation (7.13) we have:

$$T_a * (T_b * T_c) = \partial X_{-1}(\frac{\partial}{\partial t_a}) \circ \partial X_{-1}(\frac{\partial}{\partial t_b}) \circ \partial X_{-1}(\frac{\partial}{\partial t_c})\, T_0$$

Therefore, on account of the symmetry condition $\partial X_{-1} \wedge \partial X_{-1} = 0$,

$$\begin{aligned} T_a * (T_b * T_c) &= \partial X_{-1}(\frac{\partial}{\partial t_a}) \circ \partial X_{-1}(\frac{\partial}{\partial t_b}) \circ \partial X_{-1}(\frac{\partial}{\partial t_c})\, T_0 \\ &= \partial X_{-1}(\frac{\partial}{\partial t_c}) \circ \partial X_{-1}(\frac{\partial}{\partial t_a}) \circ \partial X_{-1}(\frac{\partial}{\partial t_b})\, T_0 \\ &= T_c \cup (T_a * T_b) = (T_a * T_b) \cup T_c = (T_a * T_b) * T_c \end{aligned}$$

since the binary operation $*$ defined by equation (7.22) agrees with cup product except on classes of degree 2.

To verify that this construction is compatible with passage to the germ of $\mathcal{V}$, observe that if $\tilde{\mathcal{V}}$ is variation of Hodge structure which is governed by equation (7.13) and agrees with $\mathcal{V}$ over some neighborhood O of zero in D_σ then

$$\Phi|_O = \tilde{\Phi}|_O$$

To establish the converse, let Φ be a holomorphic solution of the WDVV equation which satisfies conditions (7.18) and (7.19). Then, because the potential function Φ is an honest holomorphic function and not merely a formal power series, we may use conditions (7.18) and (7.19) to define a candidate Hodge filtration $\mathcal{F}$ over some neighborhood of zero in D_σ via equation (7.13).

Likewise, we may use the Dubrovin connection ∇ defined by equation (7.8) and our potential function Φ to define a candidate for the integral structure $\mathcal{V}_{\mathbb{Z}}$ and the polarization Q.

Now, as the reader will see upon reviewing the results of §6, in order to verify that the triple $(\mathcal{V}_{\mathbb{Z}}, \mathcal{F}, Q)$ defines a variation of pure, polarized Hodge structure of weight 3 over some neighborhood of zero in D_σ, it will suffice to check the following three conditions:

(1) The Higgs field $\partial X_{-1} = \Omega + \partial\Gamma_{-1}$ defined by equation (7.13) satisfies the symmetry condition $\partial X_{-1} \wedge \partial X_{-1} = 0$.

(2) The function $\Gamma = \Gamma_{-1} + \Gamma_{-2} + \Gamma_{-3}$ constructed by Theorem (6.16) from the Higgs field ∂X_{-1} is compatible with the given definitions of Γ_{-2} and Γ_{-3} appearing in equation (7.13).

(3) The triple $(\mathcal{V}_{\mathbb{Z}}, \mathcal{F}, Q)$ defines a polarized Hodge structure at each point q of some neighborhood of zero in D_σ.

To verify condition (1), one first recalls the commutativity and associativity properties of the product $*$ and then simply computes the action of the two form $\partial X_{-1} \wedge \partial X_{-1}$ on $\mathcal{V}$.

To verify condition (2), one simply checks that the functions Γ_{-2} and Γ_{-3} defined by equation (7.13) are a solution of the system of differential equations

$$\begin{aligned} \partial\Gamma_{-2} &= [\Gamma_{-1}, \Omega + \frac{1}{2}\partial\Gamma_{-1}] \\ \partial\Gamma_{-3} &= [\Gamma_{-2}, \Omega + \frac{1}{2}\partial\Gamma_{-1}] + \frac{1}{12}[\Gamma_{-1}, [\Gamma_{-1}, \partial\Gamma_{-1}]] \end{aligned}$$

derived at the end of §6.

Finally, to verify condition (3), one may proceed as in [4] and simply check that the limiting mixed Hodge structure (F_∞, W) of $\mathcal{V}$ is polarized by the bilinear form Q and the monodromy cone

$$\mathcal{C} = \{\, a_1 N_1 + \cdots + a_n N_n \mid a_1, \dots, a_n > 0 \,\}$$

Now, in order to prove that the germ of $\mathcal{V}$ only depends upon the germ of Φ, observe that if $\tilde{\Phi}$ is a potential function which agrees with Φ over some

neighborhood O of zero in D_σ then

$$\mathcal{V}|_O = \tilde{\mathcal{V}}|_O$$

Remark. *In order to avoid having to work at the level of germs in the preceding theorem, observe that equation* (7.13) *also governs a Hodge–Tate variation since the Hodge filtration $\mathcal{F}$ is opposed to the monodromy weight filtration $\mathcal{W}$ over all of D_σ.*

Finally a reminder: All of the results stated in this section are predicated upon the implicit assumption that our given large radius limit point σ satisfies the integrality condition (7.3).

8 B model

We shall now revisit [6] from the viewpoint of the Higgs field carried by a Hodge–Tate variation. For reference, the rough outline of [6] is as follows:

(1) In the vicinity of a maximally unipotent boundary point, the variations of pure, polarized Hodge structure $\mathcal{H}$ arising in the B–model of mirror symmetry give rise to a corresponding variation of mixed Hodge structure $\mathcal{V} \to \Delta^{*n}$ which is obtained by pairing the Hodge filtration $\mathcal{F}$ of $\mathcal{H}$ with the monodromy weight filtration $\mathcal{W}$.

(2) The variations of mixed Hodge structure described in (1) are of Hodge-Tate type, defined over $\mathbb{Z}$, and may therefore described in terms of extension classes (sections)

$$E_* \in End_{\mathbb{Z}}(Gr^{\mathcal{W}})_{-2} \otimes \mathcal{O}^*_{mer}(\Delta^{*n}) \tag{8.1}$$

Moreover, given a generator $1 \in Gr^W_0(\mathbb{Z})$,

$$Gr^W_{-2} = \operatorname{span}_{\mathbb{Z}}(N_1(1), \dots, N_n(1)) \tag{8.2}$$

(3) By virtue of (8.1) and (8.2), there exists canonical coordinates $(q_1, \dots, q_n)$ on Δ^n relative to which $E_0 : Gr^W(\mathbb{Z}) \to Gr^W_{-2}(\mathbb{Z})$ assumes the form

$$E_0(1) = \sum_{j=1}^n q_j N_j(1) \tag{8.3}$$

(4) The Yukawa coupling of the underlying physical theory may be computed as the "logarithmic derivative" of $E_1 : Gr^{\mathcal{W}}_{-2} \to Gr^{\mathcal{W}}_{-4}$.

Now, for Hodge-Tate variation $\mathcal{V}$, the corresponding Deligne grading $\mathcal{Y}$ determines both the Hodge filtration and weight filtration of $\mathcal{V}$. Moreover the connection ∇ of $\mathcal{V}$ may be expressed as

$$\nabla = \nabla^h + \theta$$

with $\nabla^h = \bar{\partial} + \partial$ preserving each subbundle $I^{p,p}$ of $\mathcal{V}$ and

$$\theta \in \Omega^1 \otimes \mathfrak{g}^{-1,-1} \tag{8.4}$$

Therefore, as discussed in §8 of [6], the vector bundle isomorphism $\mathcal{Y} : \mathcal{V} \cong Gr^{\mathcal{W}}$ allows the transport of the flat connection ∇ of $\mathcal{V}$ to the connection

$$D = d - \frac{1}{2\pi i} \log E \tag{8.5}$$

on $Gr^{\mathcal{W}}$, where E is the extension class of $\mathcal{V}$ and d denotes the standard connection on $Gr^{\mathcal{W}}$ defined by the integral structure $Gr^{\mathcal{W}}(\mathbb{Z})$.

Unraveling definitions, direct computation shows that the connection appearing in (8.5) may be rewritten as

$$D = d + \theta$$

relative to the natural action

$$\theta : T^{hol}(\Delta^{*n}) \otimes Gr^{\mathcal{W}}_{2p} \to Gr^{\mathcal{W}}_{2p-2} \tag{8.6}$$

induced on each $Gr^{\mathcal{W}}_{2p}$ by (8.4). Comparing (8.5) and (8.6) it therefore follows that:

$$\theta = -\frac{1}{2\pi i} d \log E \tag{8.7}$$

In particular, the Yukawa coupling is given by the action

$$d \log E_{-2} = -2\pi i \theta \, : \, T^{hol}(\Delta^{*n}) \otimes Gr^{\mathcal{W}}_{-2} \to Gr^{\mathcal{W}}_{-4}$$

Remark. *In [6], $\mathcal{V}$ is a variation in homology, hence $Gr^{\mathcal{W}}_k = 0$ for $k > 0$. While I shall keep Deligne's notation, I will continue to refer to $\mathcal{V}$ in terms of cohomology.*

Suppose now that the fibers of the constant local system $Gr^{\mathcal{W}}$ arise as the middle cohomology of an algebraic variety (the mirror), or more generally a family of such varieties. Then, there must be a natural cup product structure on $Gr^{\mathcal{W}}$ as well as a notion of Poincaré duality (assuming the mirror variety has only mild singularities). As discussed by [6], these structures should be induced by the pairing[1] $Q : Gr^{W}_{-k}(\mathbb{Z}) \otimes Gr^{W}_{-6-k}(\mathbb{Z}) \to \mathbb{Z}$ determined by the

[1] Note: Q is the pairing on $Gr^{\mathcal{W}}$ obtained from the SL_2 orbit theorem. See [2] for details.

variation $\mathcal{V}$ and the algebra structure $Gr^{\mathcal{W}}(\mathbb{Z})$ inherits from the polynomial algebra $\mathbb{Z}[N_1, \dots, N_n]$.

More precisely, let

$$L = \mathrm{span}_{\mathbb{Z}}(N_1, \dots, N_n)$$

Then, by virtue of (8.2), we obtain an isomorphism

$$Gr^W \cong Z \oplus L \oplus L^\vee \oplus Z^\vee$$

via the maps

$$\begin{array}{l} k \in \mathbb{Z} \mapsto k(1) \in Gr^W_0(\mathbb{Z}) \\ N \in L \mapsto N(1) \in Gr^W_{-2}(\mathbb{Z}) \end{array}$$

and the pairing Q. Under this isomorphism, the "cup product" map $L \otimes L \to L^\vee$ is determined by the functional

$$(xy)(z) := Q(1, xyz(1))$$

To obtain a deformation of this product structure on $Gr^{\mathcal{W}}$, observe that by virtue of (6.21),

$$\operatorname*{Res}_{q_j=0}(\theta) = \frac{1}{2\pi i} N_j$$

and hence introducing the vector fields $\xi_j = 2\pi i q_j \frac{\partial}{\partial q_j}$ it follows that

$$\theta(\xi_j) = N_j + (\text{higher order terms})$$

Consequently, the product structure defined by the three point function

$$\phi(N_a, N_b, N_c) = Q(1, \theta(\xi_a) \circ \theta(\xi_b) \circ \theta(\xi_c)\, 1) \tag{8.8}$$

will be a deformation of the monodromy algebra of $Gr^{\mathcal{W}}(\mathbb{Z})$.

Theorem 8.9 *The product structure determined by* (8.8) *is commutative and associative.*

Proof. Let $\alpha \cup \beta$ denote the standard cup product on $Gr^{\mathcal{W}}$ defined previously and set

$$\alpha * \beta = \begin{cases} \alpha \cdot \beta & \text{if } \alpha, \beta \in Gr^{\mathcal{W}}_{-2} \\ \alpha \cup \beta & \text{otherwise} \end{cases}$$

where $\alpha \cdot \beta$ is defined by the requirement

$$(N_a \cdot N_b) \cup N_c = \phi(\xi_a, \xi_b, \xi_c) \tag{8.10}$$

Therefore, by virtue of the symmetry condition $\theta \wedge \theta = 0$, the value of (8.10) is invariant under the natural action of the permutation group S_3 on the labels a, b, c. Consequently,

$$(N_a \cdot N_b) \cup N_c = (N_b \cdot N_c) \cup N_a = N_a \cup (N_b \cdot N_c)$$

on account of S_3–invariance and the commutativity of the product between $Gr^{\mathcal{W}}_{-2}$ and $Gr^{\mathcal{W}}_{-4}$. Likewise, $N_a \cdot N_b = N_b \cdot N_a$ because by S_3–invariance

$$(N_a \cdot N_b) \cup N_c = (N_b \cdot N_a) \cup N_c$$

Remark. *It is not necessary to explicitly determine the coordinates* $(q_1, \ldots, q_n)$ *in order to form the quantum product via this procedure. More precisely, we only need the vector fields* $\xi_1, \ldots, \xi_n$ *which may be recovered as follows: The map*

$$\theta : T^{hol}(\Delta^{*n}) \otimes Gr^{\mathcal{W}}_0 \to Gr^{\mathcal{W}}_{-2}$$

must have an expression of the form

$$\theta(\xi)\, 1 = \sum_j N_j(1) \otimes \Omega_j(\xi)$$

relative to meromorphic 1–forms $\Omega_1, \ldots, \Omega_n$*. However, comparing* (8.3) *and* (8.7)*, it follows that* $\{\xi_j\}$ *is the dual frame of* $\{\Omega_j\}$ *up to a factor of* (-1).

A Appendix

In section I communicate an alternative proof of Theorem (5.1) due to P. Deligne. With the exception of a few minor adjustments and remarks, our presentation essentially follows [7] verbatim.

For any mixed Hodge structure (F, W), one has

(a) $F^p = \oplus_{r \geq p} I^{r,s}$.

(b) Let $\bar{I}$ be obtained by interchanging the roles of F and $\bar{F}$ in the definition of I, i.e.

$$(\bar{I})^{r,s} := \overline{I^{s,r}}$$

Then,

$$\bigoplus_{s \leq q} (\bar{I})^{r,s} = \bigoplus_k W_k \cap F^{k-q} \tag{A.1}$$

Indeed, as the preceding sum, together with a bi-index (r,s) contains all $(r', s') < (r, s)$, one may replace $\bar{I}$ by I in $(A.1)$. To prove that

$$\bigoplus_{s \leq q} I^{r,s} = \bigoplus_k W_k \cap F^{k-q} \tag{A.2}$$

observe that

(1) By definition,
$$W_k \cap F^{k-q} = \bigoplus_{r+s\leq k,\, r\geq k-q} I^{r,s}$$
In particular, $W_k \cap F^{k-q} \subseteq \bigoplus_{s\leq q} I^{r,s}$ since
$$r+s \leq k,\;\; r \geq k-q \implies r+s \leq k \leq r+q \implies s \leq q$$

(2) Conversely,
$$\bigoplus_{s\leq q} I^{r,s} \subseteq \bigoplus_{k} W_k \cap F^{k-q}$$
since given (r,s) with $s \leq q$ there exists an integer k such that
$$r+s \leq k \leq r+q$$
(i.e. $r+s \leq k$ and $r \geq k-q$).

Now, if Φ_* is the increasing filtration (b) attached to a variation of graded-polarized mixed Hodge structure $\mathcal{V}$ then Griffiths transversality for for the Hodge filtration $\mathcal{F}$ gives rise to a "transversality" condition
$$\nabla\Phi_q \subseteq \Omega^1 \otimes \Phi_{q+1} \tag{A.3}$$

For the C^∞ bundle E underlying such a variation, $\mathcal{F}$ and $\bar{\Phi}$ are the two filtrations attached to the decomposition
$$E = \bigoplus_p \mathcal{U}^p, \qquad \mathcal{U}^p = \bigoplus_q \mathcal{I}^{p,q} \tag{A.4}$$
(i.e. $\mathcal{F}^p = \bigoplus_{a\geq p} \mathcal{U}^a$ and $\overline{\Phi}_q = \bigoplus_{a\leq q} \mathcal{U}^a$.) That $\mathcal{F}$ and Φ are holomorphic and obey transversality is equivalent to the fact that the direct sum decomposition $(A.4)$ defines a complex variation of Hodge structure with respect to ∇.

Remark. *In addition to its directness, this approach has the following advantages over the proof of Theorem* (5.1) *presented in §5:*

- *It avoids the introduction of the classifying spaces of graded-polarized mixed Hodge structures $\mathcal{M}$ constructed in §4.*
- *It shows that we may drop the the assumption of graded-polarizability from the statement of Theorem* (5.1).

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